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MULTIPLIERS AND CONVOLUTORS IN THE SPACE OF TEMPERED ULTRADISTRIBUTIONS

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Abstract. We consider the space of convolutors and the space of multipliers of Beurling and Roumieu tempered ultradistributions and we give characterization theorems for them.

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1. Introduction

In [21] and [6] convolution operators and multipliers of the space $\mathcal{S}$ were studied by L. Schwartz and J. Horvath. Later, G. Sampson, Z. Zielezny, [19], [23] characterized convolution operators of the spaces $\mathcal{K}'_p$, $p \geq 1$. D. H. Pahk, [15] considered convolution operators in $\mathcal{K}'_e$. Topological structure of the spaces of multipliers and convolutors in $\mathcal{K}'_M$ was studied by S. Abdulah, [1].

H. Komatsu in [9] was the first who gave a systematic approach of the spaces of ultradistributions, using sequence (M_p), satisfying certain conditions. There is another approach with weight functions, which is used by C. Fernández, A. Galbis, M.C. Gómez - Collado, in [3], [4], [5]. The convolution in ultradistribution spaces were considered in [7] by S. Pilipović, A. Kaminski, D. Kovačević, while convolutors in the spaces of ultradistributions were investigated in [2], [7], [8], [12], [16], [17], [18].

Our main interest in this paper are convolutors and multipliers in the space of tempered ultradistributions of Beurling and Roumieu type and their characterization. To motivate the research on convolutors let us consider the following example:

Let $P(D) = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha$ (with suitable assumptions on coefficients), then the equation $P(D)u = v$ can be rewritten in the form $P(\delta) * u = v$. Hence, considering equations of the type $S * u = v$ one generalizes the concept of ultra differential operators with constant coefficients. In order to consider such equations, S must be an ultradistribution that has well-defined convolution with elements of $\mathcal{S}^{(M_p)}$ resp. $\mathcal{S}^{\{M_p\}}$.

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The paper is organized as follows: Section 2 contains notation and basic definitions. Section 3 is devoted to the space of convolutors. The novelty is that we give structure theorems for the space of convolutors in the Roumieu case, as well as the completeness of $O_C'^{(M_p)}$, resp. $O_C'^{\{M_p\}}$. In Section 4 we consider the space of multipliers $O_M^{(M_p)}$, resp. $O_M^{\{M_p\}}$. Characterization theorem for the space of multipliers in Roumieu case is given. The Fourier transform gives a topological isomorphism between the space of multipliers and the space of convolutors in Roumieu case.

2. Notation

The sets of non-negative integers, natural, real and complex numbers are denoted by $\mathbf{N}_0, \mathbf{N}, \mathbf{R}, \mathbf{C}$. We use the symbols for $x \in \mathbf{R^n}$: $\langle x\rangle = (1+ \mid x \mid^2)^{1/2}$, $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, $D_j^{\alpha_j} = i^{-1}\partial^{\alpha_j}/\partial x^{\alpha_j}$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbf{N_0^n}$. With τ_x we will denote translation by x, i.e. $\tau_x f(t) = f(t + x)$.

By M_p we denote a sequence of positive numbers. The following conditions on this sequences will be assumed (see [9]):

(M.1) (Logarithmic convexity)

$$M_p^2 \le M_{p-1}M_{p+1}, \ \ p \in \mathbf{N};$$

(M.2) (Stability under ultradifferential operators)

$$M_p \le AH^p \min_{0\le q\le p}\{M_{p-q}M_q\}, \ \ p, q \in \mathbf{N}_0, \quad \text{for some} \quad A, H \ge 0;$$

(M.3) (Strong non-quasi-analyticity)

$$\sum_{p=q+1}^{\infty} \frac{M_{p-1}}{M_p} \le Aq\frac{M_q}{M_{q+1}}, \ \ q \in \mathbf{N}.$$

(M.3)' (Non-quasi-analyticity)

$$\sum_{j=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty\,.$$

and a strictly weaker condition (It is even strictly weaker than (M.3)'):

(M.3)* (Strong non-quasi-analyticity of the square)

$$\sum_{p=q+1}^{\infty} \frac{M_{p-1}^2}{M_p^2} \le Aq\frac{M_q^2}{M_{q+1}^2}, \ \ q \in \mathbf{N}.$$

We always assume that $M_0 = 1$.

So, $M_p = p!^\sigma$, $\sigma > 1$, satisfies conditions (M.1), (M.2) and (M.3)*.

The so-called associated function for the sequence M_p is defined by

$$M(\rho) = \sup_{p\in\mathbf{N}_0}\{\log_+ \frac{\rho^p}{M_p}\}, \ \ \rho > 0.$$

In the sequel, we discuss compactly supported ultradifferentiable functions. We assume (M.1), (M.2) and (M.3)'. For the definitions and properties of the spaces $\mathcal{D}^{M_p}_{K,r}, \mathcal{D}^{(M_p)}_K, \mathcal{D}^{\{M_p\}}_K, \mathcal{D}^{(M_p)}$, $\mathcal{D}^{\{M_p\}}, \mathcal{E}^{(M_p)}, \mathcal{E}^{\{M_p\}}$ we refer to [9].

If $f \in L^1$ then its Fourier transform is defined by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbf{R^n}} e^{-ix\xi} f(x)dx, \ \ \xi \in \mathbf{R^n}.$$

By $\Re$ is denoted a set of positive sequences which increases to infinity. If $r_p \in \Re$ and K is a compact set in $\mathbf{R^n}$ then $\mathcal{D}^{\{M_p\}}_{K,r_p}$ is the space of smooth functions φ on $\mathbf{R^n}$ supported by K such that

$$\|\varphi\|_{K,r_p} = \sup\{\frac{| \varphi^{(p)}(x) |}{N_p}; \ \ p \in \mathbf{N_0^n}, \ \ x \in K\} < \infty,$$

where $N_p = M_p \prod_{i=1}^{|p|} r_i, \quad p \in \mathbf{N_0^n}$. Clearly, this is a Banach space. It is proved in [10] that

$$\mathcal{D}^{\{M_p\}}_K = \text{proj} \lim_{r_p \in \Re} \mathcal{D}^{M_p}_{K,r_p}.$$

If $\Omega \subset \mathbf{R^n}$ is a bounded open set and $r > 0$, resp. $r_p \in \Re$, we put

$$\mathcal{D}^{(M_p)}_{\Omega,r} = \text{ind} \lim_{K \subset\subset \Omega} \mathcal{D}^{M_p}_{K,r}, \quad \mathcal{D}_{\Omega,r_p} = \text{ind} \lim_{K \subset\subset \Omega} \mathcal{D}^{\{M_p\}}_{K,r_p}.$$

The associated function for the sequence N_p is

$$N_{r_p}(\rho) = \sup\{\log_+ \frac{\rho^p}{N_p}; \ \ p \in \mathbf{N}_0\}, \ \ \rho > 0.$$

Note, for given r_p and every $k > 0$ there is $\rho_0 > 0$ such that

$$N_{r_p}(\rho) \leq M(k\rho), \ \ \rho > \rho_0. \tag{2.1}$$

With conditions (M.1), (M.2) and (M.3) we define ultradifferential operators. It is said that $P(\xi) = \sum_{\alpha \in \mathbf{N_0^n}} a_\alpha \xi^\alpha, \quad \xi \in \mathbf{R^n}$, is an ultrapolynomial of the class (M_p), resp. $\{M_p\}$, whenever the coefficients a_α satisfy the estimate

$$| a_\alpha | \leq C L^\alpha M_\alpha, \ \ \alpha \in \mathbf{N_0^n}, \tag{2.2}$$

for some $L > 0$ and $C > 0$ resp. for every $L > 0$ and some $C_L > 0$. The corresponding operator $P(D) = \sum_\alpha a_\alpha D^\alpha$ is an ultradifferential operator of the class (M_p) resp $\{M_p\}$.

Assume now (M.1), (M.2) and (M.3) and put

$$P_r(\xi)=(1+\mid\xi\mid^2)\prod_{p\in\mathbf{N^n}}\Big(1+\frac{\mid\xi\mid^2}{r^2m_p^2}\Big),\quad resp. \tag{2.3}$$
$$P_{r_p}(\xi)=(1+\mid\xi\mid^2)\prod_{p\in\mathbf{N^n}}\Big(1+\frac{\mid\xi\mid^2}{r_p^2m_p^2}\Big),\ \xi\in\mathbf{C^n},$$

where $m_p=M_p/M_{p-1}$ and $r>0$ resp. $r_p\in\Re$. Conditions (M.1), (M.2) and (M.3) imply that P_r resp. P_{r_p} is an ultradifferential operator of the class (M_p) resp. of the class $\{M_p\}$ (see [9]). The following estimations

$$\mid P_r(\xi)\mid\geq e^{M(r|\xi|)},\ \ \xi\in\mathbf{R^n}, \tag{2.4}$$
$$\mid P_{r_p}(\xi)\mid\geq e^{N_{r_p}(|\xi|)},\ \ \xi\in\mathbf{R^n},$$

will be used. Assume (M.1), (M.2) and (M.3). We denote by $\mathcal{S}_2^{M_p,m}(\mathbf{R^n}),\ m>0$, the space of smooth functions φ which satisfy

$$\sigma_{m,2}(\varphi):=\Big(\sum_{p,q\in\mathbf{N_0^n}}\int_{\mathbf{R^n}}\Big|\frac{m^{p+q}\langle x\rangle^p\varphi^{(q)}(x)}{M_pM_q}\Big|^2dx\Big)^{1/2}<\infty, \tag{2.5}$$

supplied with the topology induced by the norm $\sigma_{m,2}$. If instead of 2 we put $p\in[1,\infty]$ in (2.5) one obtains the equivalent sequence of norms $\sigma_{m,p},\ m>0$.

The spaces $\mathcal{S}'^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$ of tempered ultradistributions of Beurling and Roumieu type respectively, are defined as the strong duals of the spaces

$$\mathcal{S}^{(M_p)}=\lim\text{proj}_{m\to\infty}\mathcal{S}_2^{M_p,m}(\mathbf{R^n})\quad\text{and}\quad\mathcal{S}^{\{M_p\}}=\lim\text{ind}_{m\to 0}\mathcal{S}_2^{M_p,m}(\mathbf{R^n}),$$

respectively. The common notation for symbols (M_p) and $\{M_p\}$ will be $*$.

All the good properties of $\mathcal{S}^*$ and its strong dual follow from the equivalence of the sequence of norms $\sigma_{m,p},\ m>0,$, $p\in[1,\infty]$ with the each of the following sequences of norms ([13]), ([2]) :

(a) $\sigma_{m,p},\ \ m>0,\ p\in[1,\infty]$ is fixed ;

(b) $s_{m,p},\ m>0,\ p\in[1,\infty]$ is fixed, where

$$s_{m,p}(\varphi):=\sum_{\alpha,\beta\in\mathbf{N_0^n}}\frac{m^{\alpha+\beta}\|x^\beta\varphi^{(\alpha)}\|_p}{M_\alpha M_\beta};$$

(c) $s_m,\ \ m>0$, where $s_m(\varphi):=\sup_{\alpha\in\mathbf{N_0^n}}\dfrac{m^\alpha\|\varphi^{(\alpha)}e^{M(m\cdot)}\|_{L_\infty}}{M_\alpha}$;

In [2] it is proved that

$$\mathcal{S}^{\{M_p\}}=\text{proj}\lim_{r_i,s_j\in\Re}S^{M_p}_{r_i,s_j},$$

where

$$S^{M_p}_{r_i,s_j} = \{\varphi \in C^\infty(\mathbf{R^n});\ \gamma_{r_i,s_j}(\varphi) < \infty\},$$

and

$$\gamma_{r_i,s_j}(\varphi) := \sum_{p,q\in\mathbf{N_0^n}} \{\frac{\|\langle x\rangle^p \varphi^{(q)}\|_{L^\infty}}{(\prod_{i=1}^p r_i)M_p(\prod_{j=1}^q s_j)M_q}\}.$$

Note that $\mathcal{F} : \mathcal{S}^* \to \mathcal{S}^*$ is a topological isomorphism and that the Fourier transformation on $\mathcal{S}'^*$ is defined as usually.

3. The space of convolutors

Assume (M.1), (M.2) and (M.3).

The space of the convolutors O'^*_C, of $\mathcal{S}'^*$ is the space of all $S \in \mathcal{S}'^*$ such that the convolution $S * \varphi$ is in $\mathcal{S}^*$, for every $\varphi \in \mathcal{S}^*$, and the mapping

$$\varphi \longrightarrow S * \varphi, \quad \mathcal{S}^* \longrightarrow \mathcal{S}^* \quad \text{is continuous.}$$

We recall from [17] several results.

Proposition 3.1. *If $\varphi \in \mathcal{S}^*$ and $S \in \mathcal{S}'^*$ then,*

$$(S * \varphi)(x) = \langle S(t), \varphi(x - t)\rangle, \ \ x \in \mathbf{R^n},$$

is a smooth function which satisfies the following condition:

There is $k > 0$, resp. there is $k_p \in \Re$, such that for every operator P of class $*$ and $\varphi \in \mathcal{S}^*$

(3.1)
$$P(D)(S * \varphi)(x) = O(e^{M(k|x|)}), |\, x \,|\to \infty, \quad \textit{resp.}$$
$$P(D)(S * \varphi)(x) = O(e^{N_{k_p}(|x|)}), \ \ |\, x \,|\to \infty.$$

From the definition, we have that for $S \in O'^*_C$ the mapping

$$T \to S * T, \quad \mathcal{S}'^* \to \mathcal{S}'^*, \quad \text{is continuous.}$$

Proposition 3.2. *Let $S \in \mathcal{S}'^*$. The following statements are equivalent.*

a) S is a convolutor.

b) For every $\varphi \in \mathcal{D}^*$, $S * \varphi \in \mathcal{S}^*$.

c) For every $r > 0$, resp. there exist $k > 0$

$$\{e^{M(r|x|)} S(\cdot - x);\ \ x \in \mathbf{R}\} \ \textit{resp.}$$
$$\{e^{M(k|x|)} S(\cdot - x);\ \ x \in \mathbf{R}\},$$

is bounded in $\mathcal{D}'^*$.

d) For every $r>0$, resp. there exist $k>0$, there is $l>0$, resp. there is $k_p\in\Re$, and L^∞ functions F_1 and F_2 such that

$$S=P_l(D)F_1+F_2,\quad \textit{resp.}\quad S=P_{k_p}(D)F_1+F_2,$$

and

$$\|e^{M(r|x|)}(|\,F_1(x)\,|+|\,F_2(x)\,|)\|_{L^\infty}<\infty$$

resp.

$$\|e^{M(k|x|)}(|\,F_1(x)\,|+|\,F_2(x)\,|)\|_{L^\infty}<\infty.$$

Proof. We will prove only Roumieu case. Beurling case is similar.
a) $\Rightarrow$ b) It's obvious.
b) $\Rightarrow$ c) Let $\varphi\in\mathcal{D}^*$.

$$\langle e^{M(k|x|)}\tau_x S_t,\varphi(t)\rangle=\langle e^{M(k|x|)}S_t,\varphi(t+x)\rangle=$$

$$=e^{M(k|x|)}(S*\check{\varphi})(-x)\,.$$

$$|\,e^{M(k|x|))}(S*\check{\varphi})(-x)\,|\le Cs_k(S*\check{\varphi})\,.$$

c) $\Rightarrow$ d) For this part we will need the following lemma of H. Komatsu [11].

Lemma 3.3. *Let K be a compact neighborhood of zero, $r>0$, and $r_p\in\Re$.*

i) There are $u\in\mathcal{D}^{(M_p)}_{K,r/2}$ and $\psi\in\mathcal{D}^{(M_p)}_K$ such that

$$P_r(D)u=\delta+\psi\,, \tag{3.2}$$

where P_r is of form (2.3).

ii) There are $u\in C^\infty$ and $\psi\in\mathcal{D}^{\{M_p\}}_K$ such that

$$P_{r_p}(D)u=\delta+\psi\,, \tag{3.3}$$

$$\operatorname{supp}u\subset K,\quad \sup_{x\in K}\left\{\frac{|\partial^\alpha u(x)|}{\prod_{j=1}^{|\alpha|}r_j\,M_\alpha}\right\}\longrightarrow 0,\quad |\alpha|\to\infty\,, \tag{3.4}$$

where P_{r_p} is of form (2.3).

Let Ω be a bounded open set in $\mathbf{R^n}$ which contains zero and $K=\overline{\Omega}$. Let B be a bounded set in $\mathcal{D}^{\{M_p\}}_K$. For $\varphi\in B$

$$|\,\langle e^{M(k|x|)}\tau_x S_t,\varphi(t)\rangle\,|=e^{M(k|x|)}\,|\,(S*\check{\varphi})(-x)\,|\le C\,, \tag{3.5}$$

for all $x\in\mathbf{R^n}$ where $C>0$ does not depend on $\varphi\in B$. Denote by $L^1_{exp(-M(k|\cdot|))}$ the space of locally integrable functions f on $\mathbf{R^n}$ such that $f(\cdot)e^{-M(k|\cdot|)}\in L^1(\mathbf{R^n})$. We supply this space with the norm

$$\|f\|_{L^1,exp(-M(k|\cdot|))}=\|f(\cdot)e^{-M(k|\cdot|)}\|_{L^1}\,.$$

Let B_1 be the closed unit ball in the space $L^1_{exp(-M(k|\cdot|))}$, $\psi \in B_1 \cap \mathcal{D}^{\{M_p\}}$ and $\varphi \in B$. Then,

$$(3.6) \qquad |\langle S * \psi, \varphi \rangle| = |\langle (S * \check{\varphi})(-x), \psi \rangle| \leq$$

$$\leq \|S * \check{\varphi}(-x) \cdot e^{M(k|x|)}\|_{L^\infty} \cdot \|\psi\|_{L^1, exp(-M(k|\cdot|))} \leq C\|\psi\|_{L^1, exp(-M(k|\cdot|))} \leq C\,.$$

Hence

$$(3.7) \qquad |\langle S * \psi, \varphi \rangle| \leq C\|\psi\|_{L^1, exp(-M(k|\cdot|))}$$

for all $\varphi \in B$ and $\psi \in \mathcal{D}^{\{M_p\}}$. From (3.6) it follows that

$$\{S * \psi |\ \psi \in B_1 \cap \mathcal{D}^{\{M_p\}}\}$$

is bounded set in $\mathcal{D}'^{\{M_p\}}_K$, and because $\mathcal{D}'^{\{M_p\}}_K$ is barrelled, the set is equicontinuous. There exist $k_p \in \Re$ and $\varepsilon > 0$ such that

$$|\langle S * \theta, \check{\psi} \rangle| \leq 1\,, \psi \in B_1 \cap \mathcal{D}^{\{M_p\}},\ \theta \in V_{k_p}(\varepsilon),$$

where

$$(3.8) \qquad V_{k_p}(\varepsilon) = \{\chi \in \mathcal{D}^{\{M_p\}}_K |\ \|\chi\|_{K,k_p} \leq \varepsilon\}\,.$$

The same inequality holds for the closure $\overline{V_{k_p}(\varepsilon)}$ of $V_{k_p}(\varepsilon)$ in $\mathcal{D}^{\{M_p\}}_{K,k_p}$. If $\theta \in \mathcal{D}^{\{M_p\}}_{\Omega,k_p}$, then for some $L_\theta > 0$, $\|\theta/L_\theta\|_{K,k_p} < \varepsilon$. Hence $\theta/L_\theta \in \overline{V_{k_p}(\varepsilon)}$ and $|\langle S * \theta, \check{\psi} \rangle| \leq L_\theta$, for $\psi \in B_1 \cap \mathcal{D}^{\{M_p\}}$. It follows that for $\psi \in \mathcal{D}^{\{M_p\}}$

$$(3.9) \qquad |\langle S * \theta, \check{\psi} \rangle| \leq L_\theta \|\psi\|_{L^1, exp(-M(k|\cdot|))}\,.$$

Because $\mathcal{D}^{\{M_p\}}$ is dense in $L^1_{exp(-M(k|\cdot|))}$ it follows that for every θ in $\mathcal{D}^{\{M_p\}}_{\Omega,k_p}$, $S * \theta$ is a continuous functional on $L^1_{exp(-M(k|\cdot|))}$. Thus $S * \theta$ belongs to $L^\infty_{exp(M(k|\cdot|))} = \{f \in L_{1,loc} \,|\, \|f(\cdot)e^{M(k|\cdot|)}\|_{L_\infty} < \infty\}$, since the space $L^\infty_{exp(M(k|\cdot|))}$ is the dual of the space $L^1_{exp(-M(k|\cdot|))}$. Hence,

$$\|S * \theta(x)\|_{L^\infty, exp(M(k|\cdot|))} \leq L_\theta\,,$$

where $L_\theta > 0$ is a constant which depends of θ. From Lemma 3.3 for the chosen $k_p \in \Re$ and Ω there exist $\tilde{k_p}$ and $u \in \mathcal{D}^{\{M_p\}}_{\Omega,k_p}$ and $\psi \in \mathcal{D}^{\{M_p\}}_\Omega$ such that

$$S = P_{\tilde{k_p}}(D)(u * S) + (\psi * S)\,.$$

Now it's obvious that $F_1 = u * S$ and $F_2 = \psi * S$ satisfy the conditions in d).
d) $\Rightarrow$ a) We will assume that $F_2 = 0$. The general case is proved analogously. It is enough to prove that $\varphi \longrightarrow F * \varphi$ is a continuous mapping from $\mathcal{S}^{\{M_p\}}$ to $\mathcal{S}^{\{M_p\}}$. Then a) will hold because of the continuity of the operator $P_{k_p}(D)$ and the fact that $P_{k_p}(D)(S * \varphi) = P_{k_p}(D)S * \varphi$. Let us observe that the continuity of the mapping $\varphi \longrightarrow F * \varphi$ will follow if we prove that for every r which is

bigger than some fixed r_0, there exist l such that $\varphi \longrightarrow F*\varphi$ is a continuous mapping from $\mathcal{S}_\infty^{M_p,r}$ to $\mathcal{S}_\infty^{M_p,l}$ (because $\mathcal{S}^{\{M_p\}}$ is a inductive limit of $\mathcal{S}_\infty^{M_p,r}$). For the k in the condition d) we choose r_0, small enough such that for all $r \leq r_0$ the integral

$$\int_{\mathbf{R^n}} e^{-M(k|t|)} e^{M(r|t|)} dt$$

converges. We fix r such that $r \leq r_0$. Note that

$$\frac{r^p \mid x \mid^p}{2^p M_p} \leq \frac{r^p \mid x-t \mid^p}{M_p} + \frac{r^p \mid t \mid^p}{M_p} \leq \tag{3.10}$$

$$\leq e^{M(r|x-t|)} + e^{M(r|t|)} \leq 2e^{M(r|x-t|)} e^{M(r|t|)}$$

and the last inequality holds since the function $M(\rho)$ is nonnegative. For the associated function there exist $\rho_0 > 0$ such that for $\rho \leq \rho_0$, $M(\rho) = 0$ and for $\rho > \rho_0$, $M(\rho) > 0$ (for the properties of the associated function we refer to [9]). If $\mid x \mid > \dfrac{2\rho_0}{r}$ then from the inequality (3.10) it follows that

$$e^{M(\frac{r}{2}|x|)} \leq 2e^{M(r|x-t|)} e^{M(r|t|)} .$$

If $\mid x \mid \leq \dfrac{2\rho_0}{r}$, there exist $c > 0$ such that $e^{M(\frac{r}{2}|x|)} \leq c$. Hence, it follows that for all $x \in \mathbf{R^n}$, the following inequality holds

$$e^{M(\frac{r}{2}|x|)} \leq 2(c+1)\, e^{M(r|x-t|)} e^{M(r|t|)}$$

and we get

$$e^{-M(r|x-t|)} \leq C e^{M(r|t|)} e^{-M(\frac{r}{2}|x|)} ,$$

where we put $C = 2(c+1)$. Let $l < r/4$. Then,

$$\frac{l^\alpha \mid F * D^\alpha \varphi(x) \mid e^{M(l|x|)}}{M_\alpha} \leq \frac{l^\alpha}{M_\alpha} \int_{\mathbf{R^n}} \mid F(t) \mid\mid D^\alpha \varphi(x-t) \mid dt\, e^{M(l|x|)} =$$

$$= \left(\frac{l}{r}\right)^\alpha \frac{1}{M_\alpha} \int_{\mathbf{R^n}} \mid F(t) \mid \frac{e^{M(k|t|)}}{e^{M(k|t|)}} \mid D^\alpha \varphi(x-t) \mid r^\alpha \frac{e^{M(r|x-t|)}}{e^{M(r|x-t|)}} dt\, e^{M(l|x|)} \leq$$

$$\leq C' \left(\frac{l}{r}\right)^\alpha s_r(\varphi) \int_{\mathbf{R^n}} e^{-M(k|t|)} e^{M(r|t|)} dt\, e^{-M(\frac{r}{2}|x|)}\, e^{M(l|x|)}.$$

Because of the way we choose l, it follows that

$$s_l(F*\varphi) = \sup_\alpha \frac{l^\alpha \| \mid F * D^\alpha \varphi(x) \mid e^{M(l|x|)} \|_{L^\infty}}{M_\alpha} \leq C'' s_r(\varphi) ,$$

where C'' is a constant which does not depend on φ. We have shown that $\varphi \longrightarrow F*\varphi$ is a continuous mapping from $\mathcal{S}_\infty^{M_p,r}$ to $\mathcal{S}_\infty^{M_p,l}$. Hence, $\varphi \longrightarrow F*\varphi$ is a continuous mapping from $\mathcal{S}^{\{M_p\}}$ to $\mathcal{S}^{\{M_p\}}$. □

It is clear that the ultratempered convolution of $S_1, S_2 \in O'^*_C$ is in O'^*_C (see [7]). Also for any $T \in \mathcal{S}'^*$, and $\psi \in \mathcal{S}^*$ we have

$$\begin{aligned} \langle (S_1 * S_2) * T, \psi\rangle &= \langle S_1 * T, \check{S}_2 * \psi\rangle = \\ &= \langle T * S_2, \check{S}_1 * \psi\rangle = \langle T, (S_1 * S_2) * \psi\rangle \,. \end{aligned} \tag{3.11}$$

We supply O'^*_C with the topology from $L_{\mathrm{s}}(\mathcal{S}^*, \mathcal{S}^*)$ and denote it by $O'^*_{C,\mathrm{s}}$. The same topology on this space is induced by $L_{\mathrm{s}}(\mathcal{S}'^*, \mathcal{S}'^*)$.

Proposition 3.4. *The strong topology on $L(\mathcal{S}'^*, \mathcal{S}'^*)$ induces the same topology on O'^*_C.*

Proof. Let U be a neighborhood of zero in $\mathcal{S}'^*$. Without loss of generality we can assume that

$$U = U(V'; B') = \{S \in O'^*_C(\mathcal{S}'^*; \mathcal{S}'^*) \mid S * T \in V' \,, for\, all\; T \in B'\} \,,$$

where B' is bounded subset in $\mathcal{S}'^*$ and V' is a neighborhood of zero in $\mathcal{S}'^*$. We can assume that

$$V' = V'(B, \varepsilon) = \{T \in \mathcal{S}'^* \mid \mid \langle T, \varphi\rangle \mid < \varepsilon \, for\, all\, \varphi \in B\} \,,$$

where B is bounded in $\mathcal{S}^*$, and $\varepsilon > 0$. Let

$$V = \{\varphi \in \mathcal{S}^* \mid \mid \langle T, \varphi\rangle \mid < \varepsilon \, for\, all\, T \in B'\} \,.$$

Since $\mathcal{S}^*$ is barreled is follows that V is a neighborhood of zero in $\mathcal{S}^*$. Without loss of generality we will assume that $B = \check{B} = \{\check{\varphi} \mid \varphi \in B\}$ and $B' = \check{B}' = \{\check{T} \mid T \in B'\}$. Let

$$W = W(V, B) = \{S \in O'^*_C(\mathcal{S}'^*; \mathcal{S}'^*) \mid S * \varphi \in V \; for\, all\, \varphi \in B\} \,.$$

We will show that $W(V, B) \subset U(V', B')$. Let $S \in W(V, B)$, $T \in B'$ and $\varphi \in B$. Then

$$\mid \langle S * T, \varphi\rangle \mid = \mid \langle T, \check{S} * \varphi\rangle \mid < \varepsilon \,.$$

Hence $S * T \in V'$ for all $T \in B'$. We have shown that the topology induced by $L_s(\mathcal{S}'^*, \mathcal{S}'^*)$ is stronger than the topology induced by $L_s(\mathcal{S}^*, \mathcal{S}^*)$. The other direction is similar and will be omitted. □

Proposition 3.5. *$O'^*_{C,s}$ is complete.*

Proof. Let $\{S_\mu\}$ be a Cauchy net in $O'^*_{C,s}$. Then $\{\tilde{S}_\mu\}$ is a Cauchy net in $L_s(\mathcal{S}^*, \mathcal{S}^*)$, where $\tilde{S}_\mu : \mathcal{S}^* \longrightarrow \mathcal{S}^*$ are induced continuous linear operators by S_μ, $\tilde{S}_\mu(\varphi) = S_\mu * \varphi$. Since $\mathcal{S}^*$ is complete and bornological [22], Corollary 1 of Theorem 32.2, $L_s(\mathcal{S}^*, \mathcal{S}^*)$ is complete, there exists $R \in L_s(\mathcal{S}^*, \mathcal{S}^*)$, such that $\tilde{S}_\mu \longrightarrow R$. We define $T \in \mathcal{S}'^*$ by $\langle T, \varphi\rangle = R(\varphi)(0)$. For $\varphi \in \mathcal{S}^*$, $R(\varphi) = T * \varphi$, since for $x \in \mathbf{R^n}$

$$R(\varphi)(x) = \lim_\mu (S_\mu * \varphi)(x) = \lim_\mu (S_\mu * (\tau_x \varphi))(0) =$$

$$= R(\tau_x\varphi)(0) = \langle T, \tau_x\varphi\rangle = T * \varphi(x)\,.$$

Thus for $\varphi \in \mathcal{S}^*$, $T * \varphi \in \mathcal{S}^*$ and the map $\varphi \longrightarrow T * \varphi$ is continuous. It follows that $T \in O'^*_C$, and moreover $S_\mu \longrightarrow T$ in O'^*_C since $\tilde{T} = R$. □

Proposition 3.6. *A sequence S_n from $O'^*_{C,s}$ converges to zero in $O'^*_{C,\mathrm{s}}$ if and only if for every $k > 0$ resp. there exist $k > 0$, there exists $r > 0$, resp. there exists $k_p \in \Re$ and sequences of L^∞ functions F_{1n} and F_{2n}, such that*

$$S_n = P_r(D)F_{1n} + F_{2n}, \quad \textit{resp. } S_n = P_{k_p}(D)F_{1n} + F_{2n}, \tag{3.12}$$

$F_{1n}, F_{2n} \in O'^*_C$,

$$\|e^{M(k|x|)}(| F_{1n} | + | F_{2n} |)\|_{L^\infty} < \infty$$

and

$$F_{1n} \longrightarrow 0, \quad F_{2n} \longrightarrow 0 \qquad \textit{in } O'^*_{C,s}\,, \tag{3.13}$$

Proof. The proof of the proposition is similar to the proof of the Proposition 3.2, but we will give it for completeness. Let S_n be a sequence in $O'^{\{M_p\}}_C$ which converges to zero in $O'^{\{M_p\}}_{C,\mathrm{s}}$. Let Ω be a bounded open set in $\mathbf{R^n}$ which contains zero and $K = \overline{\Omega}$. Let $\varphi \in \mathcal{D}^{\{M_p\}}_K$ be fixed. Then $S_n * \varphi \longrightarrow 0$ in $\mathcal{S}^{\{M_p\}}$. Because $\mathcal{S}^{\{M_p\}}$ is a (DFS) space, it follows that there exist $k > 0$ such that $S_n * \varphi \in \mathcal{S}^{M_p,k}_\infty$, and is bounded there, i.e.

$$\sup_\alpha \frac{k^\alpha \|e^{M(k|x|)} D^\alpha (S_n * \varphi)(x)\|_{L^\infty}}{M_\alpha} \le C_\varphi, \forall n \in \mathbf{N}\,,$$

where C_φ is a constant which depends only on φ. We get

$$\|e^{M(k|x|)}(S_n * \varphi)(x)\|_{L^\infty} \le C_\varphi\,, \forall n \in \mathbf{N}\,.$$

Let $\psi \in B_1 \cap \mathcal{D}^{\{M_p\}}$, then

$$| \langle S_n * \psi, \check{\varphi}\rangle | = | \langle S_n * \varphi, \check{\psi}\rangle | \le \|S_n * \varphi\|_{L^\infty_{exp(M(k|\cdot|))}} \le C_\varphi\,, \tag{3.14}$$

for all $n \in \mathbf{N}$, where B_1 is the closed unit ball in $L^1_{exp(-M(k|\cdot|))}$.

From (3.14) it follows that

$$\{S_n * \psi |\ \psi \in B_1 \cap \mathcal{D}^{\{M_p\}}, n \in \mathbf{N}\}$$

is weakly bounded set in $\mathcal{D}'^{\{M_p\}}_K$, and because $\mathcal{D}^{\{M_p\}}_K$ is barrelled, the set is equicontinuous (see [20], Theorem 5.2). There exist $k_p \in \Re$ and $\delta > 0$ such that

$$| \langle S_n * \theta, \check{\psi}\rangle | \le 1\,,\ \theta \in V_{k_p}(\delta)\,, \psi \in B_1 \cap \mathcal{D}^{\{M_p\}}\,,\ n \in \mathbf{N},$$

where $V_{k_p}(\delta) = \{\chi \in \mathcal{D}^{\{M_p\}}_K |\, \|\chi\|_{K,k_p} \le \delta\}$. The same inequality holds for the closure $\overline{V_{k_p}(\delta)}$ of $V_{k_p}(\delta)$ in $\mathcal{D}^{\{M_p\}}_{K,k_p}$. If $\theta \in \mathcal{D}^{\{M_p\}}_{\Omega,k_p}$, then for some $L_\theta > 0$, $\|\theta/L_\theta\|_{K,k_p} < \delta$, hence $\theta/L_\theta \in \overline{V_{k_p}(\delta)}$ and

$$| \langle S_n * \theta, \check{\psi}\rangle | \le L_\theta\,,\ \psi \in B_1 \cap \mathcal{D}^{\{M_p\}}\,,\ n \in \mathbf{N}\,.$$

It follows that for $\psi \in \mathcal{D}^{\{M_p\}}$

$$| \langle S_n * \theta, \check{\psi} \rangle | \leq L_\theta \|\psi\|_{L^1, exp(-M(k|\cdot|))} . \tag{3.15}$$

Because $\mathcal{D}^{\{M_p\}}$ is dense in $L^1_{exp(-M(k|\cdot|))}$, it follows that for every θ in $\mathcal{D}^{\{M_p\}}_{\Omega,k_p}$, $S_n * \theta$ are continuous functionals on $L^1_{exp(-M(k|\cdot|))}$ and uniformly bounded. Thus $S_n * \theta$ belong to $L^\infty_{exp(M(k|\cdot|))}$. Hence,

$$\|S_n * \theta(x)\|_{L^\infty, exp(M(k|\cdot|))} \leq L_\theta \, , \forall n \in \mathbf{N} \, ,$$

where $L_\theta > 0$ is a constant which depends on θ. From Lemma 3.3, for the chosen $k_p \in \Re$ and Ω, there exist $\tilde{k_p}$ and $u \in \mathcal{D}^{\{M_p\}}_{\Omega,k_p}$ and $\psi \in \mathcal{D}^{\{M_p\}}_{\Omega}$ such that

$$S_n = P_{\tilde{k_p}}(D)(S_n * u) + (S_n * \psi) \, .$$

Let $F_{1n} = S_n * u$ and $F_{2n} = S_n * \psi$. It's obvious that $u \in O'^{\{M_p\}}_C$, hence $F_{1n}, F_{2n} \in O'^{\{M_p\}}_C$. $F_{1n} = S_n * u \longrightarrow 0$ and $F_{2n} = S_n * \psi \longrightarrow 0$ in $O'^{\{M_p\}}_C$.

Conversely, let $F_n \longrightarrow 0$ and $F_{1n} \longrightarrow 0$ in $O'^{\{M_p\}}_C$, $S_n = P_{k_p}(D)F_n + F_{1n}$, for some $k_p \in \Re$. We will assume that $F_{1n} = 0$ for all $n \in \mathbf{N}$. The general case is proved similarly. Let $M(B,V)$ be a neighborhood of zero in $O'^{\{M_p\}}_C$, where B is a bounded set in $\mathcal{S}^{\{M_p\}}$, and V is a open neighborhood of zero in $\mathcal{S}^{\{M_p\}}$. Since, $P_{k_p}(D) : \mathcal{S}^{\{M_p\}} \longrightarrow \mathcal{S}^{\{M_p\}}$ is continuous, there exists an open neighborhood V_0 such that $P_{k_p}(D)(V_0) \subset V$. Since $F_n \longrightarrow 0$ in $O'^{\{M_p\}}_C$, and $M(B,V_0)$ is a neighborhood of zero, there exists n_0, such that for all $n \geq n_0$, $F_n \in M(B,V_0)$. Thus, $F_n * \varphi \in V_0$, for all $\varphi \in B$ and $n \geq n_0$, and it follows that

$$P_{k_p}(D)(F_n * \varphi) \subset P_{k_p}(D)(V_0) \subset V \, .$$

□

Remark 3.7. The inclusion $O'^*_{C,s} \hookrightarrow \mathcal{S}'^*$ is continuous. Let V be a open neighborhood in $\mathcal{S}'^*$. Let us consider this neighborhood of O'^*_C:

$$W = \{S \in O'^*_C \,|\, S * \delta \in V\} \, .$$

Then it is obvious that from $S \in W$ follows that $S \in V$. We get that, from the convergence of F_{1n}, F_{2n} to zero in O'^*_C in the above proposition follows the convergence in $\mathcal{S}'^*$.

We denote by $\mathcal{ES}'^*$ the space of elements f from $\mathcal{S}'^*$ such that for every $S \in O'^*_C$, $S * f \in \mathcal{E}^*$ and the mapping

$$S \to S * f, \;\; O'^*_{C,\mathrm{s}} \to \mathcal{E}^* \text{ is continuous.}$$

Proposition 3.8. *(i)* $\mathcal{ES}'^* \subset \mathcal{E}^* \cap \mathcal{S}'^*$.

(ii) If $f \in \mathcal{ES}'^*$ *and* $S \in O'^*_C$ *then* $S * f \in \mathcal{ES}'^*$.

Proof. (i) It's clear from the definition of $\mathcal{E}\mathcal{S}'^*$ that if $f \in \mathcal{E}\mathcal{S}'^*$, $f \in \mathcal{S}'^*$ and because δ is in $O_C'^{\{M_p\}}$ we obtain $\delta * f = f$ is an element in $\mathcal{E}^*$.

(ii) From (i) it follows that $S * f \in \mathcal{S}'^*$. Let $T \in O_C'^*$. We have that

$$T * (S * f) = (T * S) * f$$

is in $\mathcal{E}^*$. It's obvious that the mapping $T \longrightarrow T * (S * f)$ is continuous, since the mappings $T \longrightarrow T * S \longrightarrow (T * S) * f = T * (S * f)$ are continuous. Hence, $S * f \in \mathcal{E}\mathcal{S}'^*$. □

Note that $\mathcal{S}^*$ is subset of $\mathcal{E}\mathcal{S}'^*$.

4. The space of multipliers

Again we assume (M.1), (M.2) and (M.3).

As in [12] and [16] , we define O_M^* as the space of functions φ from $\mathcal{E}^*$ such that $\varphi \in O_M^*$ if and only if

$$\text{(4.1)} \qquad \begin{array}{c} \text{for every } \psi \in \mathcal{S}^*,\ \varphi\psi \in \mathcal{S}^* \quad \text{and the mapping} \\ \psi \to \varphi\psi, \quad \mathcal{S}^* \longrightarrow \mathcal{S}^* \quad \text{is continuous .} \end{array}$$

From the definition, we have that for $\varphi \in O_M^*$ the mapping

$$T \to \varphi T, \quad \mathcal{S}'^* \to \mathcal{S}'^*, \quad \text{is continuous.}$$

In the proof of the next proposition we will need the following function:

$$\text{(4.2)} \qquad \psi(x) = \sum_{j=1}^{\infty} \frac{\rho(x - x_j)}{e^{M(k|x_j|)}},$$

where the function $\rho \in \mathcal{D}^{\{M_p\}}$, with values in $[0, 1]$, such that

$$\operatorname{supp}\rho \subset \{x \mid |x| \leq 1, x \in \mathbf{R^n}\},\ \rho(\mathbf{x}) = \mathbf{1},$$

for $x \in \{x \mid\ | x |\leq 1/2\}$, and $\{x_j\}$ is a sequence of real numbers such that $| x_j |> 2$ and $| x_{j+1} |\geq| x_j | +2$, $j \in \mathbf{N}$.

Since $\rho \in \mathcal{D}^{\{M_p\}}$, there exist h and C such that $\sup\limits_{x,\alpha} | D^\alpha \rho |< Ch^\alpha M_\alpha$. We will show that $\psi \in \mathcal{S}^{\{M_p\}}$. We choose r such that $rh < \dfrac{1}{2}$ and $r < \dfrac{k}{H4\sqrt{2}}$. Using that $\dfrac{| x |}{| x_j |} \leq 2$, we have,

$$\sum_{\alpha,\beta} \int\limits_{\mathbf{R^n}} \frac{r^{2\alpha+2\beta}\langle x\rangle^{2\beta} \mid D^\alpha\psi(x) \mid^2}{M_\alpha^2 M_\beta^2} dx$$

$$\leq \sum_{\alpha,\beta}\sum_{j=1}^{\infty}\int_{|x-x_j|\leq 1}\frac{r^{2\alpha+2\beta}\langle x\rangle^{2\beta}C^2h^{2\alpha}M_\alpha^2}{M_\alpha^2M_\beta^2e^{2M(k|x_j|)}}dx$$

$$\leq \sum_{\alpha,\beta}\sum_{j=1}^{\infty}\int_{|x-x_j|\leq 1}\frac{r^{2\alpha+2\beta}\langle x\rangle^{2\beta}C^2h^{2\alpha}}{M_\beta^2e^{2M(k|x_j|)}}dx$$

$$\leq \sum_{\alpha,\beta}\sum_{j=1}^{\infty}\int_{|x-x_j|\leq 1}\frac{r^{2\alpha}r^{2\beta}2^{\beta}\mid x\mid^{2\beta}C^2h^{2\alpha}}{M_\beta^2e^{2M(k|x_j|)}}dx$$

$$\leq C_1\sum_{\alpha,\beta}\sum_{j=1}^{\infty}\frac{(rh)^{2\alpha}(r\sqrt{2})^{2\beta}}{M_\beta^2e^{2M(k|x_j|)}}\int_{|x-x_j|\leq 1}\mid x\mid^{2\beta}dx$$

$$\leq C_2\sum_{\alpha,\beta}\sum_{j=1}^{\infty}\frac{(rh)^{2\alpha}(r\sqrt{2})^{2\beta}\mid x_j\mid^{2\beta}}{M_\beta^2e^{2M(k|x_j|)}}$$

$$\leq C_2\sum_{\alpha,\beta}\sum_{j=1}^{\infty}\frac{(rh)^{2\alpha}(r2\sqrt{2})^{2\beta}\mid x_j\mid^{2\beta}M_{\beta+1}^2}{M_\beta^2k^{2\beta+2}\mid x_j\mid^{2\beta+2}}$$

$$\leq \frac{C_2A}{k^2}\sum_{\alpha,\beta}\sum_{j=1}^{\infty}(rh)^{2\alpha}\Big(\frac{2r\sqrt{2}H}{k}\Big)^{2\beta}\frac{1}{\mid x_j\mid^2}\leq C'\,.$$

The proof of the next proposition in (M_p)-case is given in [12] and [16].

Proposition 4.1. *Let $\varphi\in C^\infty$. The following statements are equivalent.*

(i) $\varphi\in O_M^*$.

(ii) For every $h>0$, resp. for every $k>0$, there exist $k>0$, resp. there exist $h>0$,

$$\sup_{\alpha\in\mathbf{N_0^n}}\{\frac{h^\alpha\|e^{-M(k|\cdot|)}\varphi^{(\alpha)}\|_{L^\infty}}{M_\alpha}\}<\infty.$$

(iii) For every $\psi\in\mathcal{S}^*$ and every $r>0$, resp. for some $r>0$.

$$\sigma_{m,\psi}(\varphi):=\sigma_{m,\infty}(\psi\varphi)<\infty.$$

(iv) In Roumieu case, for every $\psi\in\mathcal{S}^{\{M_p\}}$ and for every $r_i,s_j\in\Re$

$$\gamma_{r_i,s_j,\psi}(\varphi):=\gamma_{r_i,s_j}(\psi\varphi)<\infty.$$

Proof. We will give the proof only for the Roumieu case.

(iii)⇔ (iv) it's obvious. We will prove (iii)⇒ (ii)⇒(i)⇒(iii).

(iii)⇒(ii) First we will prove that φ is in $\mathcal{E}^{\{M_p\}}$. Let K be a fixed compact set in $\mathbf{R^n}$ and take $\chi\in\mathcal{D}^{\{M_p\}}$, with values in $[0,1]$ and $\chi(x)=1$ on a neighborhood of K. Then there exist r such that

$$\sup_{\alpha}\frac{r^\alpha\|D^\alpha(\varphi(x)\chi(x))\|_{L^\infty(K)}}{M_\alpha}\leq$$

$$\leq \sup_{\alpha} \frac{r^{\alpha}\|e^{M(r|x|)}D^{\alpha}(\varphi(x)\chi(x))\|_{L^{\infty}(\mathbf{R^n})}}{M_{\alpha}} = Cs_r(\varphi\chi) < \infty\,.$$

We have $D^{\alpha}(\varphi(x)\chi(x)) = D^{\alpha}\varphi(x)$ for $x \in K$. Thus $\varphi \in \mathcal{E}^{\{M_p\}}$.

Let (ii) does not hold. Then there exist k such that for all $n \in \mathbf{N}$,

$$\sup_{\alpha} \frac{\|e^{-M(k|x|)}D^{\alpha}\varphi(x)\|_{L^{\infty}}}{n^{\alpha}M_{\alpha}} = \infty\,.$$

Since $\varphi \in \mathcal{E}^{\{M_p\}}$ for every compact set K, there exist C and $n_K \in \mathbf{N}$ such that for $n \geq n_K$

$$\sup_{\alpha} \frac{\|e^{-M(k|x|)}D^{\alpha}(\varphi(x))\|_{L^{\infty}(K)}}{n^{\alpha}M_{\alpha}} < C\, n \geq n_K\,.$$

Hence, we can choose α_j and x_j, where $| x_{j+1} |>| x_j | +2$, such that

$$\frac{e^{-M(k|x_j|)} \mid D^{\alpha_j}\varphi(x_j) \mid}{j^{\alpha_j}M_{\alpha_j}} \geq 1\,.$$

Now take ψ as in (4.2), where we take k and the sequence $\{x_j\}$ to be the ones chosen here. Then $\varphi\psi \in \mathcal{S}^{\{M_p\}}$, i.e. there exist l such that

$$\sup_{\alpha} \frac{l^{\alpha}\|e^{M(k|x|)}D^{\alpha}(\varphi(x)\psi(x))\|_{L^{\infty}}}{M_{\alpha}} < \infty\,.$$

Then there exist j_0 such that for all $j \geq j_0$, $l > 1/j$.

$$\sup_{\alpha} \frac{l^{\alpha}\|e^{M(l|x|)}D^{\alpha}(\varphi(x)\psi(x))\|_{L^{\infty}}}{M_{\alpha}} \geq$$

$$\geq \frac{l^{\alpha_j}e^{M(l|x_j|)} \mid D^{\alpha_j}(\varphi(x_j)\psi(x_j)) \mid}{M_{\alpha_j}} \geq$$

$$\geq \frac{1}{j^{\alpha_j}} \frac{e^{M(l|x_j|)} \mid D^{\alpha_j}\varphi(x_j) \mid}{e^{M(k|x_j|)}M_{\alpha_j}} \geq e^{M(l|x_j|)}\,.$$

This implies that $\varphi\psi$ is not in $\mathcal{S}_{\infty}^{M_p,l}$, which is a contradiction with the above assumption.

(ii)$\Rightarrow$(i) From the condition (ii) it is obvious that $\varphi \in \mathcal{E}^{\{M_p\}}$. It is enough to prove that for every $r > 0$ there is $l > 0$ such that the mapping $\psi \longrightarrow \varphi\psi$ for $\mathcal{S}_{\infty}^{M_p,r}$ to $\mathcal{S}_{\infty}^{M_p,l}$ is continuous. Let $r > 0$ be fixed. Put $k = r/4$. By (ii), there exist h such that

$$\sup_{\alpha} \frac{h^{\alpha}\|e^{-M(k|x|)}D^{\alpha}\varphi(x)\|_{L^{\infty}}}{M_{\alpha}} < \infty\,.$$

If $l < h/4$ and $l < r/4$, then

$$\frac{l^{\alpha}\|e^{M(l|x|)}D^{\alpha}(\varphi(x)\psi(x))\|_{L^{\infty}}}{M_{\alpha}} \leq$$

$$\leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\frac{l^{\alpha}\|e^{M(l|x|)}D^{\beta}\varphi(x)D^{\alpha-\beta}\psi(x)\|_{L^{\infty}}}{M_{\alpha-\beta}M_{\beta}}=$$

$$=\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\frac{(2l)^{\alpha}\|e^{M(l|x|)}D^{\beta}\varphi(x)e^{-M(k|x|)}e^{M(k|x|)}h^{\beta}}{2^{\alpha}h^{\beta}}\cdot$$

$$\cdot\frac{D^{\alpha-\beta}\psi(x)e^{M(r|x|)}e^{-M(r|x|)}r^{\alpha-\beta}\|_{L^{\infty}}}{r^{\alpha-\beta}M_{\alpha-\beta}M_{\beta}}\leq$$

$$\leq Cs_r(\psi)\|e^{-M(r|x|)}e^{M(k|x|)}e^{M(l|x|)}\|_{L^{\infty}}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\frac{1}{2^{\alpha}}\Big(\frac{2l}{h}\Big)^{\beta}\Big(\frac{2l}{r}\Big)^{\alpha-\beta}\leq C's_r(\psi),$$

where the last inequality holds because of the way we choose l.

(i)⇒(iii) it's obvious. □

Remark 4.2. It's obvious that if $\varphi\in O_M^*$, then $\varphi\in\mathcal{S}'^*$.

Denote by $L(\mathcal{S}^*,\mathcal{S}^*)$ the space of continuous linear mappings from $\mathcal{S}^*$ into $\mathcal{S}^*$; O_M^* is its subspace. With $L_{\mathrm{s}}(\mathcal{S}^*,\mathcal{S}^*)$ we denote the space $L(\mathcal{S}^*,\mathcal{S}^*)$ with the strong topology. We can also equip O_M^* with the topology induced by $L_{\mathrm{s}}(\mathcal{S}'^*,\mathcal{S}'^*)$. Similarly as in Proposition 3.4 we can prove that the topologies induced by $L_{\mathrm{s}}(\mathcal{S}^*,\mathcal{S}^*)$ and $L_{\mathrm{s}}(\mathcal{S}'^*,\mathcal{S}'^*)$ are the same. The space O_M^* equipped with this topology is denoted by $O_{M,\mathrm{s}}^*$.

Proposition 4.3. *The Fourier transformation is a topological isomorphism of* $O_{M,\mathrm{s}}^*$ *onto* $O_{C,\mathrm{s}}'^*$.

Proof. We only show the Roumieu case. Using Proposition 3.2 d), there exist $k>0$ and there exist $k_p\in\Re$ such that $S=P_{k_p}(D)F+F_1$, where F and F_1 satisfy the growth condition given in Proposition 3.2. Without loss of generality we may assume that $F_1=0$. By (M.2) we obtain the following estimates for the derivatives of the Fourier transform of F:

$$|\,D^{\alpha}\mathcal{F}(F)\,|=|\,\mathcal{F}(x^{\alpha}F)\,|=\Big|\int_{\mathbf{R^n}}e^{-ix\xi}x^{\alpha}F(x)dx\Big|\leq \tag{4.3}$$

$$\leq\int_{\mathbf{R^n}}|\,x\,|^{\alpha}\,|\,F(x)\,|\,dx\leq C_1\int_{\mathbf{R^n}}|\,x\,|^{\alpha}\,e^{-M(k|x|)}dx\leq$$

$$\leq C_2\int_{\mathbf{R^n}}\frac{|x|^{\alpha}}{\langle x\rangle^{\alpha+n+1}}M_{\alpha+n+1}\Big(\frac{c}{k}\Big)^{|\alpha|+n+1}dx\leq CM_{\alpha}M_{n+1}\Big(\frac{Hc}{k}\Big)^{|\alpha|+n+1}.$$

In ([9]), page 88, the following estimate of the analytic function $P_{k_p}(\zeta)$ is given: For every L there is C such that

$$|\,P_{k_p}(\zeta)\,|\leq ACe^{M(\sqrt{n}LH|\zeta|)},\zeta\in\mathbf{C^n}.$$

Using this and the Cauchy integral formula, we obtain that for every $L>0$ there exist $C>0$ such that

$$|\,D^{\alpha}P_{k_p}(\xi)\,|\leq C\frac{\alpha!}{d^{\alpha}}\cdot e^{M(Lc'|\xi|)}, \tag{4.4}$$

where $c' > 0$ is a constant that does not depend on L. It is also known that, for every $m > 0$,

$$\frac{m^k k!}{M_k} \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty \tag{4.5}$$

Let $m > 0$ be arbitrary. Let L be a constant such that

$$\|e^{-M(m|\xi|)}e^{M(Lc'|\xi|)}\|_{L^\infty} < \infty\,,$$

and h is chosen such that $2h < 1$ and $2hHc < k$. From (4.3), (4.4), (4.5) and (M.1) we obtain,

$$\begin{aligned}
&\sup_\alpha \frac{h^\alpha \|e^{-M(m|\xi|)}D^\alpha(P_{r_p}(\xi)\hat{F}(\xi))\|_{L^\infty}}{M_\alpha}\\
&\leq \sup_\alpha \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\frac{(2h)^\alpha\|e^{-M(m|\xi|)}D^{\alpha-\beta}P_{r_p}(\xi)D^\beta\hat{F}(\xi)\|_{L^\infty}}{2^\alpha M_{\alpha-\beta}M_\beta}\\
&\leq C\sup_\alpha \frac{1}{2^\alpha}\|e^{-M(m|\xi|)}e^{M(Lc'|\xi|)}\|_{L^\infty}\cdot\\
&\quad\cdot\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\frac{(\alpha-\beta)!}{M_{\alpha-\beta}d^{\alpha-\beta}}M_{n+1}(2h)^\beta\left(\frac{Hc}{k}\right)^{|\beta|+n+1}\\
&\leq C_1\sup_\alpha \|e^{-M(m|\xi|)}e^{M(Lc'|\xi|)}\|_{L^\infty}\frac{1}{2^\alpha}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\left(\frac{2hHc}{k}\right)^{|\beta|}\\
&\leq C_2\|e^{-M(m|\xi|)}e^{M(Lc'|\xi|)}\|_{L^\infty}\leq C_3\,.
\end{aligned}$$

By Proposition 4.1 (ii), it follows that $\hat{S} \in O_M^{\{M_p\}}$ and it is obvious that the mapping $S \longrightarrow \hat{S}$ is injective.

Now we will prove that the Fourier transform from $O_M^{\{M_p\}}$ to $O_C'^{\{M_p\}}$ is an injective mapping. Let $\varphi \in O_M^{\{M_p\}}$ and $\psi \in \mathcal{S}^{\{M_p\}}$. The mappings

$$\hat{\psi} \longrightarrow \psi \longrightarrow \varphi\psi \longrightarrow \mathcal{F}(\varphi\psi) = \left(\frac{1}{2\pi}\right)^n \hat{\varphi} * \hat{\psi}$$

are continuous from $\mathcal{S}^{\{M_p\}}$ to $\mathcal{S}^{\{M_p\}}$. Hence, $\hat{\varphi} \in O_C'^{\{M_p\}}$ and the mapping $\varphi \longrightarrow \hat{\varphi}$ is injective from $O_M^{\{M_p\}}$ into $O_C'^{\{M_p\}}$. Now it's enough to see that the same things hold for the $\bar{\mathcal{F}} = (2\pi)^n\mathcal{F}^{-1}$ and the fact that $\mathcal{F}$ is isomorphism on $\mathcal{S}^{\{M_p\}}$ and $\mathcal{S}'^{\{M_p\}}$ with an inverse $\mathcal{F}^{-1}$. Because $\mathcal{F} : \mathcal{S}^* \longrightarrow \mathcal{S}^*$ is a topological isomorphism it's obvious that it is also a topological isomorphism from $O^*_{M,\mathrm{s}}$ to $O'^*_{C,\mathrm{s}}$. □

Proposition 4.4. *The bilinear mappings*

$$O^*_{M,s} \times \mathcal{S}^* \to \mathcal{S}^*,\;\; (\alpha,\psi) \to \alpha\psi,$$

$$O^*_{M,s} \times \mathcal{S}'^* \to \mathcal{S}'^*,\;\; (\alpha,f) \to \alpha f,$$

are hypocontinuous.

Proof. It is obvious that the bilinear mappings are separately continuous. We will prove only that the mapping $T : O^*_{M,s} \times \mathcal{S}^* \longrightarrow \mathcal{S}^*$, defined by $T(\varphi, \psi) = \varphi\psi$ is hypocontinuous. Since $\mathcal{S}^*$ is barrelled space, from [20] Theorem 5.2., it follows that for every open set V in $\mathcal{S}^*$, and every bounded set B in $O^*_{M,s}$, then there is an open set W in $\mathcal{S}^*$ such that $T(B \times W) \subset V$. Now, let V_1 is arbitrary open set in $\mathcal{S}^*$ and let B_1 be a bounded set in $\mathcal{S}^*$. Then, for the open set W_1 in $O^*_{M,s}$, where $W_1 = \{\psi \in O^*_{M,s} \,|\, \varphi\psi \in V,\ for\, all\ \psi \in B\}$, we have $T(W_1 \times B_1) \subset V_1$. □

Proposition 4.5. *The space* $O^*_{M,\mathrm{s}}$ *is nuclear.*

Proof. Since the space $\mathcal{S}^*$ is reflexive and the space S'^* is nuclear, from [20] it follows that $L_s(\mathcal{S}^*, \mathcal{S}^*)$ is nuclear space. Thus, the space $O^*_{M,\mathrm{s}}$ is nuclear as a subspace of a nuclear space. □

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