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Canonical bilaterally commutative groupoids

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Abstract A groupoid $\boldsymbol{G}$ is called bilaterally commutative if it satisfies the identities $(xy)z \approx (yx)z$ and $x(yz) \approx x(zy)$. A description of free objects in the variety $\mathcal{B}_c$ of bilaterally commutative groupoids and their characterization by means of injective objects in $\mathcal{B}_c$ is presented.

Keywords Variety of groupoids · Free groupoid · injective groupoid

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1 Introduction and preliminaries

Ježek and Kepka in [5] studied varieties of groupoids (a groupoid being an algebra with one binary operation) defined by a set of linear identities with length $\leqslant 6$. They obtained that there are 16 such non-equivalent linear identities and investigated some of the groupoid varieties, denoted by $\mathcal{V}_0, \ldots, \mathcal{V}_{15}$, determined only by one of those 16 identities. It was also proved that there are exactly 56 groupoid varieties, denoted by $\mathcal{V}_0, \ldots, \mathcal{V}_{55}$, defined by a set of linear identities with length $\leqslant 6$. Among them is the groupoid variety, denoted by $\mathcal{V}_{28}$, that is equal to the intersection of the variety $\mathcal{V}_5$, defined by the identity $x(yz) \approx x(zy)$, and its dual variety $\mathcal{V}_{13}$, defined by the identity $(xy)z \approx (yx)z$. Clearly, every commutative groupoid satisfies both of the identities, but the converse is not true.

In [5] free objects in the variety $\mathcal{V}_5$ (and its dual $\mathcal{V}_{13}$) are obtained using a free commutative groupoid over a non-empty set X. In this paper we give a description of free objects in the groupoid variety $\mathcal{V}_{28}$ (which we call the variety of bilaterally commutative groupoids) using the term groupoid $\mathbb{T}_X = (T_X, \cdot)$ over a non-empty set X. Instead of $\mathcal{V}_{28}$ we denote it by $\mathcal{B}_c$.

Throughout the paper the terms are denoted by $t, u, v, w \ldots$ We use the notation $P(t)$ for the set of subterms of a term t, $var(t)$ for the set of all variables which occur in

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t and $|t|$ for the length of t. The term groupoid $\mathbb{T}_X$ is an absolutely free groupoid over X and it is *injective*, i.e. the operation $\cdot$ is an injective mapping: $tu = vw \Rightarrow t = v,\ u = w$. The set X is the set of primes in $\mathbb{T}_X$ that generates $\mathbb{T}_X$. (An element a of a groupoid $\boldsymbol{G} = (G, \cdot)$ is said to be *prime* in $\boldsymbol{G}$ if $a \neq xy$, for all $x, y \in G$.) These two properties of $\mathbb{T}_X$ characterize all absolutely free groupoids.

Proposition 1.1 ([1], Lemma 1.5) *A groupoid $\boldsymbol{H} = (H, \cdot)$ is an absolutely free groupoid if and only if it satisfies the following two conditions:*

(i) *$\boldsymbol{H}$ is injective;*
(ii) *The set of primes in $\boldsymbol{H}$ is nonempty and generates $\boldsymbol{H}$.*

Then the set of primes is the unique free generating set for $\boldsymbol{H}$.

We refer to this proposition as *Bruck Theorem* for the class of all groupoids.

We will use the shortlex ordering of terms (denoted by $\leq$), i.e. terms are ordered so that a term of a particular length comes before any longer term, and amongst terms of equal length, lexicographical ordering is used, with terms earlier in the lexicographical ordering coming first. Then T_X is a totally ordered set. A term t is said to be *order-regular* if $t \in X$ or ($t = t_1t_2 \in T_X \setminus X$ and $t_1 \leq t_2$); otherwise, it is *order-irregular* (see [2]).

For the notation and basic notions of universal algebra the reader is referred to [4]. In most cases, without mention, the operation is denoted multiplicatively: the product of two elements x, y of a groupoid is denoted by $x \cdot y$ or just xy. The product xx is denoted by x^2 and is called the *square* of x.

2 A construction of canonical bilaterally commutative groupoids

In any variety $\mathcal{V}$ of groupoids, a groupoid $\boldsymbol{R} = (R, *)$ is called a *canonical groupoid in* $\mathcal{V}$ (or $\mathcal{V}$-*canonical groupoid*) (see [3]) if the following conditions are satisfied:

(C0) $X \subseteq R \subseteq T_X$
(C1) $tu \in R \Rightarrow t, u \in R \ \wedge \ t * u = tu$
(C2) $\boldsymbol{R}$ is a $\mathcal{V}$-free groupoid over X.

Thus, $\boldsymbol{R} = (R, *)$ is called a *canonical bilaterally commutative groupoid* if the conditions (C0), (C1) and (C2), with $\mathcal{V}$ replaced by $\mathcal{B}_c$, are satisfied.

In order to define such a groupoid, we define the carrier set R by:

$$R = \{t \in T_X : \text{ every subterm } u \text{ of } t,\ u \neq t, \text{ is order-regular}\}. \tag{2.1}$$

Obviously,

a) $X \subseteq R \subseteq T_X$; $\quad t \in R \Rightarrow P(t) \subseteq R$;
b) $t, u \in R \Rightarrow [tu \notin R \Leftrightarrow t$ is order-irregular or u is order-irregular];
c) $t, u \in T_X \Rightarrow [tu \in R \Leftrightarrow t, u \in R$ and t, u are order-regular].

To define a convenient operation on R that satisfies the conditions (C1) and (C2), we consider the four possibilities for $t, u \in R$: 1) t and u are order-regular, 2) t is order-regular and $u = u_1u_2$ is order-irregular, 3) $t = t_1t_2$ is order-irregular and u is order-regular, 4) $t = t_1t_2$ and $u = u_1u_2$ are order-irregular. We define an operation $*$ on R in the following way. For every $t, u \in R$,

$$t * u = \begin{cases} tu, & \text{if } tu \in R, \\ t(u_2u_1), & \text{if } t \text{ is order-regular} \\ & \text{and } u = u_1u_2, \quad u_2 < u_1, \\ (t_2t_1)u, & \text{if } t = t_1t_2, \ t_2 < t_1 \\ & \text{and } u \text{ is order-regular}, \\ (t_2t_1)(u_2u_1), & \text{if } t = t_1t_2, \ t_2 < t_1 \\ & \text{and } u = u_1u_2, \ u_2 < u_1. \end{cases} \tag{2.2}$$

It is clear that $t*u$ is a uniquely determined element in R, and therefore $\boldsymbol{R} = (R, *)$ is a groupoid. The definition of $*$ can be stated by an unary operation ${}^\bullet : T_X \to T_X$, defined by:

$$t \in T_X \ \Rightarrow \ t^\bullet = \begin{cases} t, & \text{if } t \in X \\ t_2t_1, & \text{if } t = t_1t_2. \end{cases} \tag{2.3}$$

We call this operation a *reversion* in $\mathbb{T}_X$. We say that $t^\bullet$ is the *reversed term* of t. Obviously, $t^\bullet = t$ if and only if $t \in X$ or t is a square in $\mathbb{T}_X$. Also, $(t^\bullet)^\bullet = t$, for every $t \in T_X$, i.e. ${}^\bullet$ is an involutory transformation on T_X. By c) it follows that $t^\bullet \in R$, for every $t \in R$. Therefore,

$$t * u = \begin{cases} tu, & \text{if } tu \in R, \\ tu^\bullet, & \text{if } t \text{ is order-regular}, \ u \text{ is order-irregular}, \\ t^\bullet u, & \text{if } t \text{ is order-irregular}, \ u \text{ is order-regular}, \\ t^\bullet u^\bullet, & \text{if } t \text{ and } u \text{ are order-irregular}. \end{cases} \tag{2.4}$$

The definition of $*$ can be written using the following transformation $\rho : R \to R$, which we call *regulation* in R, defined by:

$$t \in R \ \Rightarrow \ \rho(t) = \begin{cases} t, & \text{if } t \text{ is order-regular}, \\ t_2t_1, & \text{if } t = t_1t_2 \text{ and } t_2 < t_1. \end{cases}$$

Lemma 2.1 *If $t, u \in R$, then:*
a) $\rho(\,\rho(t)\,) = \rho(t)$, *b)* $\rho(\,\rho(t)\rho(u)\,) = \rho(\,\rho(u)\rho(t)\,)$.

Proof. a) It is clear.
b) Four cases for $t, u \in R$ are possible:
1) t, u are order-regular;
2) t is order-regular and u is order-irregular;
3) t is order-irregular and u is order-regular;
4) t, u are order-irregular.

In the first case, the equality is obvious when $t = u$. Let $t \neq u$. Then $\rho(\,\rho(t)\rho(u)\,) = \rho(tu)$ and $\rho(\,\rho(u)\rho(t)\,) = \rho(ut)$. Here, $\rho(tu) = tu = \rho(ut)$ if $t < u$ and $\rho(ut) = ut = \rho(tu)$, if $u < t$, and thus the equality holds. The fourth case is brought down to the first case, since $\rho(t), \rho(u)$ are order-regular when t, u are order-irregular. In the second case the equality is obvious when $t = \rho(u)$. Let $t \neq \rho(u)$. If $t < \rho(u)$, then: $\rho(\,\rho(t)\rho(u)\,) = \rho(\,t\rho(u)\,) = t\rho(u) = \rho(\rho(u)t) = \rho(\,\rho(u)\rho(t)\,)$. The proof goes symmetrically for $\rho(u) < t$. One can show the third case similarly as the second one. □

The operation $*$ can be presented by:

$$t, u \in R \;\;\Rightarrow\;\; t * u = \rho(t)\rho(u). \tag{2.5}$$

By Lemma 2.1b), it follows that $\boldsymbol{R} = (R, *) \in \mathcal{B}_c$. Namely, $(t * u) * v = \rho(\,\rho(t)\rho(u)\,)\,\rho(v) = \rho(\,\rho(u)\rho(t)\,)\,\rho(v) = (u*t)*v$ and $t*(u*v) = \rho(t)\,\rho(\,\rho(u)\rho(v)\,) = \rho(t)\,\rho(\,\rho(v)\rho(u)\,) = t*(v*u)$.

By induction on the length of $t \in R$ one can show that X is a generating set for $\boldsymbol{R}$. The set of prime elements in $\boldsymbol{R}$ coincides with the set X. Namely, if $t \in X$, then $t \neq u * v$, for every $u, v \in R$. If $t \in R \setminus X$, then there are $t_1, t_2 \in R$, such that $t = t_1t_2 = t_1 * t_2$. Therefore, there are no other prime elements in $\boldsymbol{R}$ except those in X.

The groupoid $\boldsymbol{R}$ has the universal mapping property for $\mathcal{B}_c$ over X, i.e. if $\boldsymbol{G} \in \mathcal{B}_c$ and $\lambda : X \to G$, then there is a homomorphism ψ from $\boldsymbol{R}$ into $\boldsymbol{G}$ such that $\psi(x) = \lambda(x)$, for every $x \in X$. We define the mapping $\psi : R \to G$ by $\psi(t) = \varphi(t)$, for every $t \in R$, where φ is the homomorphism from $\mathbb{T}_X$ into $\boldsymbol{G}$ that is an extension of λ. It suffices to show that $\varphi(t * u) = \varphi(t)\varphi(u)$, for any $t, u \in R$.

If t, u are order-regular, then $\varphi(t * u) = \varphi(tu) = \varphi(t)\varphi(u)$.

If t is order-regular and u is order-irregular, then $u = u_1u_2$ and $u_2 < u_1$, so $\varphi(t*u) = \varphi(t\rho(u_1u_2)) = \varphi(t)\varphi(u_2u_1) = \varphi(t)(\varphi(u_2)\varphi(u_1)) = [\boldsymbol{G} \in \mathcal{B}_c] = \varphi(t)(\varphi(u_1)\varphi(u_2)) = \varphi(t)\varphi(u_1u_2) = \varphi(t)\varphi(u)$.

The case when t is order-irregular and u is order-regular can be shown similarly as the previous one.

If t, u are order-irregular, then $t = t_1t_2$, $t_2 < t_1$ and $u = u_1u_2$, $u_2 < u_1$, so $\varphi(t * u) = \varphi(\rho(t_1t_2)\rho(u_1u_2)) = \varphi(t_2t_1)\varphi(u_2u_1) = (\varphi(t_2)\varphi(t_1))(\varphi(u_2)\varphi(u_1)) = [\boldsymbol{G} \in \mathcal{B}_c] = (\varphi(t_1)\varphi(t_2))(\varphi(u_1)\varphi(u_2)) = \varphi(t)\varphi(u)$.

Hence, $\boldsymbol{R} = (R, *)$ satisfies the conditions (C0), (C1) and (C2). Thus, the following theorem holds.

Theorem 2.2 *The groupoid* $\boldsymbol{R}=(R, *)$ *is a canonical bilaterally commutative groupoid over* X.

We note that $(R, *)$ is neither left nor right cancellative groupoid. For instance, if $a, b \in X$ and $t = a$, $u = ab$, $v = ba$, then $t * u = a(ab) = a * (ba) = t * v$, but $u \neq v$.

3 A characterization of free bilaterally commutative groupoids

In this section we define a subclass of the class $\mathcal{B}_c$ that is larger than the class of $\mathcal{B}_c$-free groupoids, called the class of $\mathcal{B}_c$-injective groupoids. For defining such a class we essentially use the properties of the corresponding canonical groupoid $\boldsymbol{R} = (R, *)$ in $\mathcal{B}_c$ related to the elements on $\boldsymbol{R}$ that are not prime. The class of $\mathcal{B}_c$-injective groupoids will be successfully defined if the following two conditions are satisfied. Firstly, the class of $\mathcal{B}_c$-injective groupoids should unable the characterization of $\mathcal{B}_c$-free groupoids analogously as in Prop.1.1: any $\mathcal{B}_c$-injective groupoid $\boldsymbol{H}$ whose set of primes is nonempty and generates $\boldsymbol{H}$ to be $\mathcal{B}_c$-free. Secondly, the class of $\mathcal{B}_c$-free groupoids has to be a proper subclass of the class of $\mathcal{B}_c$-injective groupoids.

Before introducing the notion of $\mathcal{B}_c$-injective groupoid it is necessary to investigate some properties of the already constructed $\mathcal{B}_c$-canonical groupoid $\boldsymbol{R}$. Every property of $\boldsymbol{R}$, whose formulation does not contain the notion prime element, is a "candidate-axiom" for $\mathcal{B}_c$-injectivity. For instance, for every non-prime element $t \in R$ there is a uniquely determined pair of divisors u, v in $\boldsymbol{R}$, i.e. t can be presented in a unique way as $t = u * v$. Other presentations of $t = x * y$, i.e. other pairs of divisors $(x, y) \in R \times R$ of t are concequences of the identities in the given variety.

Proposition 3.1 *For every non-prime element $t \in R$ there is a uniquely determined pair $(u, v) \in R \times R$ such that $t = uv = u * v = u * v^{\bullet} = u^{\bullet} * v = u^{\bullet} * v^{\bullet}$.*

Proof. Let $t \in R \setminus X$. Then $t \in T_X \setminus X$, so there are $u, v \in T_X$, such that $t = uv$, and they are uniquely determined because of the injectivity of $\mathbb{T}_X$. Since $uv = t \in R$, it follows that u, v are order-regular, and thus, by (2.4), $uv = u * v$. If u and v are primes or squares, then $u^{\bullet} = u$ and $v^{\bullet} = v$. Then, $u * v^{\bullet}$, $u^{\bullet} * v$ and $u^{\bullet} * v^{\bullet}$ are equal to $u * v$. If u is non-prime and not a square, since u is order-regular, then $u^{\bullet}$ is order-irregular and $u^{\bullet} \in R$. The same is true for v: $v^{\bullet}$ is order-irregular and $v^{\bullet} \in R$. By (2.4) and (2.3), it follows that $u * v^{\bullet} = u\,(v^{\bullet})^{\bullet} = uv = t$, $u^{\bullet} * v = (u^{\bullet})^{\bullet}\, v = uv = t$ and $u^{\bullet} * v^{\bullet} = (u^{\bullet})^{\bullet}\,(v^{\bullet})^{\bullet} = uv = t$. □

The notion of reversion can be introduced for an arbitrary groupoid, too.

Definition 3.2 *A groupoid $\boldsymbol{G} = (G, \cdot)$ is said to be* reversible *if there is a transformation $^{\bullet} : G \to G$ defined by:*

$$a \in G \;\Rightarrow\; a^{\bullet} = \begin{cases} a, & \text{if } a \text{ is prime in } \boldsymbol{G} \\ cb, & \text{if } a = bc. \end{cases} \tag{3.1}$$

In that case we say that the unary operation $^{\bullet}$ is a reversion *on $\boldsymbol{G}$.*

A characterization of reversible groupoids is given by the following

Proposition 3.3 *A groupoid $\boldsymbol{G}$ is reversible if and only if*

$$(\forall a, b, c, d \in G)\; (ab = cd \;\Rightarrow\; ba = dc).$$

Proof. Let $\boldsymbol{G}$ be a reversible groupoid and let $ab = cd$. Then $ba = (ab)^{\bullet} = (cd)^{\bullet} = dc$. Conversely, let (p, q) be a par of divisors of a, i.e. $a = pq$. Then $a^{\bullet} = qp$. Assume that (x, y) is another pair of divisors of a, i.e. $a = xy$. Then, $pq = xy$ implies that $qp = yx$. Hence, $a^{\bullet} = qp = yx$, i.e. $^{\bullet}$ is a well defined mapping. Thus, $\boldsymbol{G}$ is reversible. □

Proposition 3.4 *Any groupoid $\boldsymbol{G}$ has at most one reversion. If a reversion exists, then it is an involution (and hence a permutation) on G.*

Proof. Let f and g be reversions on $\boldsymbol{G}$ and $a \in G$. If a is prime in $\boldsymbol{G}$, then $f(a) = g(a)$. If $a = bc$, then $f(a) = cb$. Let $a = xy$ and $g(a) = yx$. Since $a = bc = xy$, it follows that $f(a) = f(bc) = f(xy) = yx = g(xy) = g(a)$. Thus, if a reversion on $\boldsymbol{G}$ exists, then it is unique. The reversion f is an involution. Namely, if a is prime, then $f(f(a)) = f(a) = a$, and if $a = bc$, then $f(f(a)) = f(cb) = bc = a$. □

Example 3.1 It is clear that every commutative groupoid is reversible. The groupoids $\boldsymbol{G}_1 = (G_1, \cdot)$, $\boldsymbol{G}_2 = (G_2, \cdot)$, $\boldsymbol{G}_3 = (G_3, \cdot)$ and $\boldsymbol{G}_4 = (G_4, \cdot)$ given below with the tables a), b), c) and d) respectively, are as follows: $\boldsymbol{G}_1$ is not reversible, since $ba = bb \neq ab$; $\boldsymbol{G}_2$ is reversible that is not commutative and is not bilaterally commutative; $\boldsymbol{G}_3$ is reversible and bilaterally commutative groupoid that is not commutative; $\boldsymbol{G}_4$ is a bilaterally commutative groupoid that is not commutative and is not reversible, since $ab = bb = c$, but $ba = d \neq bb = c$.

a)

$\cdot$	a	b
a	a	a
b	b	b

b)

$\cdot$	a	b	c
a	a	c	b
b	b	a	b
c	c	c	a

c)

$\cdot$	a	b	c	d
a	a	c	a	a
b	d	b	a	a
c	a	a	a	a
d	a	a	a	a

d)

$\cdot$	a	b	c	d
a	a	c	a	a
b	d	c	a	a
c	a	a	a	a
d	a	a	a	a

e) The following groupoid, constructed similarly as in [5], gives an example of an infinite bilaterally commutative groupoid that is not reversible. Let f, g be two surjective endomorphisms on a commutative group $(G, +)$. Define a new operation $\cdot$ on G by:

$$(\forall a, b \in G) \;\; a \cdot b = f(a) + g(b).$$

If $f \neq g$, then the obtained groupoid $(G, \cdot)$ is not commutative.

Note that: if $fg = f^2$, then the equality $(a \cdot b) \cdot c = (b \cdot a) \cdot c$ holds, and, if $gf = g^2$, then the equality $a \cdot (b \cdot c) = a \cdot (c \cdot b)$ holds, for all $a, b, c \in G$. So, if both equalities $fg = f^2$ and $gf = g^2$ hold, then the groupoid $(G, \cdot)$ is bilaterally commutative.

Let $(G, +)$ be the free commutative group over the infinite set $\{x_1, x_2, \ldots, x_n, \ldots\}$ and let f, g be defined by: $f(x_1) = f(x_2) = g(x_1) = g(x_2) = g(x_3) = x_1$, $f(x_3) = x_2$ and $f(x_i) = g(x_i) = x_{i-2}$ for $i \geq 4$. Then, $(G, \cdot)$ is a bilaterally commutative groupoid that is not commutative and is not a semigroup. By Prop.3.3, $(G, \cdot)$ is not reversible: $x_1x_3 = f(x_1) + g(x_3) = x_1 + x_1 = f(x_2) + g(x_1) = x_2x_1$, but $x_3x_1 = f(x_3) + g(x_1) = x_2 + x_1 \neq x_1 + x_1 = f(x_1) + g(x_2) = x_1x_2$.

Proposition 3.5 *If $\boldsymbol{G} = (G, \cdot)$ is a bilaterally commutative and reversible groupoid, then $ab = ab^{\bullet} = a^{\bullet}b = a^{\bullet}b^{\bullet}$, for any $a, b \in G$.*

Proof. If a, b are prime elements in $\boldsymbol{G}$, then $a^{\bullet} = a$ and $b^{\bullet} = b$. If a is prime in $\boldsymbol{G}$ and $b = cd$, then $ab = a(cd) = a(dc) = ab^{\bullet}$. It is clear that $ab = a^{\bullet}b = a^{\bullet}b^{\bullet}$. One can prove the case when $a = cd$ and b is prime in $\boldsymbol{G}$, symmetrically. If $a = cd$, $b = c'd'$, then $ab = a(c'd') = a(d'c') = ab^{\bullet}$, $ab = (cd)b = (dc)b = a^{\bullet}b$ and $ab = (cd)(c'd') = (cd)(d'c') = (dc)(d'c') = a^{\bullet}b^{\bullet}$. $\square$

For the groupoid $\boldsymbol{R} = (R, *)$ the transformation $^{\odot}$, defined by

$$t \in R \;\Rightarrow\; t^{\odot} = \begin{cases} t, & \text{if } t \in X \\ v * u, & \text{if } t = u * v. \end{cases} \tag{3.2}$$

is a reversion of $\boldsymbol{R}$. Let $u, v \in R$. If $uv \in R$, then $(u * v)^{\odot} = (uv)^{\bullet}$. If $uv \notin R$, then $(u * v)^{\odot} \neq (uv)^{\bullet}$. For instance, if u is order-regular and $v = v_1v_2$ is order-irregular, then $uv, vu \notin R$ and $(u * v)^{\odot} = v * u = (v_1v_2) * u = (v_2v_1)\, u \neq (v_1v_2)\, u = vu = (uv)^{\bullet}$.

For every $u \in R$, $t = u * u^{\odot}$ is a square. Namely, if u is order-regular, then $u * u^{\odot} = u * u = u^2$, and if $u = u_1u_2$ is order-irregular, then $u * u^{\odot} = (u_1u_2) * (u_2u_1) =$

$(u_2u_1)(u_2u_1) = (u^{\bullet})^2$. If t is non-prime in $\boldsymbol{R}$, then $t^{\odot} = t$ if and only if the divisors of t in $\boldsymbol{R}$ are equal or one of the divisors is a reversed non-prime non-square of the other. Thus, the following proposition holds.

Proposition 3.6 *Let $\boldsymbol{R} = (R, *)$ be the $\mathcal{B}_c$-canonical groupoid over X and let $u, v \in R$. Then:*

a) *$\boldsymbol{R}$ is reversible.*
b) *If $uv \in R$, then $(u * v)^{\odot} = (uv)^{\bullet}$, and if $uv \notin R$, then $(u * v)^{\odot} \neq (uv)^{\bullet}$.*
c) *$t = u*u^{\odot}$ is a square: $t = u^2$ if u is order-regular, or, $t = (u^{\bullet})^2$ if u is order-irregular.*
d) *If t is non-prime in $\boldsymbol{R}$, then $t^{\odot} = t$ if and only if t is a square.*

We will use these properties for introducing the notion of $\mathcal{B}_c$-injective groupoid.

Definition 3.7 *A groupoid $\boldsymbol{H} = (H, \cdot)$ is called* $\mathcal{B}_c$-injective *if it satisfies the following conditions:*

(0) $\boldsymbol{H} \in \mathcal{B}_c$.
(1) *$\boldsymbol{H}$ is reversible.*
(2) *If z is a non-prime element in $\boldsymbol{H}$ and (c, d) is a pair of divisors of z (i.e. $z = cd$), then: $z = ab \;\Rightarrow\; (a, b) \in \{(c, d), (c, d^{\bullet}), (c^{\bullet}, d), (c^{\bullet}, d^{\bullet})\}$.*
(3) *If z is a non-prime element in $\boldsymbol{H}$, then $z^{\bullet} = z$ if and only if z is a square.*

By Proposition 3.6, the groupoid $\boldsymbol{R} = (R, *)$ satisfies the conditions $(0) - (3)$. Since every $\mathcal{B}_c$-free groupoid over X is isomorphic with $\boldsymbol{R}$, it follows that

Proposition 3.8 *The class of $\mathcal{B}_c$-free groupoids is a subclass of the class of $\mathcal{B}_c$-injective groupoids.*

Theorem 3.9 (Bruck Theorem for $\mathcal{B}_c$) *A groupoid $\boldsymbol{H} = (H, \cdot)$ is $\mathcal{B}_c$-free if and only if it satisfies the following conditions:*

(i) *$\boldsymbol{H}$ is $\mathcal{B}_c$-injective.*
(ii) *The set P of primes in $\boldsymbol{H}$ is nonempty and generates $\boldsymbol{H}$.*

Proof. If $\boldsymbol{H}$ is $\mathcal{B}_c$-free over X, then by Prop.3.8, $\boldsymbol{H}$ is $\mathcal{B}_c$-injective. By the fact that $\boldsymbol{H}$ is isomorphic to $\mathcal{B}_c$-canonical groupoid $\boldsymbol{R}$ and the proof of Theorem 2.2, it follows that X is the set of primes in $\boldsymbol{H}$ that generates $\boldsymbol{H}$.

Let (i) and (ii) hold. It suffices to show that $\boldsymbol{H}$ has the universal mapping property for $\mathcal{B}_c$ over P. For that purpose, put $P_0 = P$, $P_{k+1} = P_k \cup P_kP_k$ and define an infinite sequence of subsets $C_0, C_1, \ldots$ of $\boldsymbol{H}$ by:

$$C_0 = P, C_1 = C_0C_0, \ldots,$$
$$C_{k+1} = \{a \in H \setminus P : (c, d)|\, a \Rightarrow \{c, d\} \subset C_0 \cup \cdots \cup C_k \;\wedge\; \{c, d\} \cap C_k \neq \emptyset\}.$$

Then, one can show (similarly as in Lemma 4.2 in [2]) that $P_k = C_0 \cup C_1 \cup \cdots \cup C_k$ and $H = \cup\{C_k : k \geq 0\}$, where $C_k \neq \varnothing$ for any $k \geq 0$ and $C_i \cap C_j = \varnothing$, for $i \neq j$.

Let $\boldsymbol{G} \in \mathcal{B}_c$ and $\lambda : P \to G$ is a mapping. Using the fact that H is a disjoint union of the sets C_k, define a sequence of mappings $\varphi_k : P_k \to G$ by: $\varphi_0(x) = \lambda(x)$ for every $x \in P$ and φ_k to be an extension of φ_{k-1} for every $k \geq 1$. Assume that $\varphi_k : P_k \to G$ is a well defined mapping. Define a mapping $\varphi_{k+1} : P_{k+1} \to G$ by:

$$\varphi_{k+1}(x) = \begin{cases} \varphi_k(x), & \text{if } x \in P_k, \\ \varphi_k(u)\varphi_k(v), & \text{if } x = uv \in C_{k+1}. \end{cases}$$

To show that φ_{k+1} is a well defined mapping, it suffices to take $x \in C_{k+1}$. Since $\boldsymbol{H}$ is $\mathcal{B}_c$-injective, x can be presented as $x = uv = u^{\bullet}v = uv^{\bullet} = u^{\bullet}v^{\bullet}$. Let's take, for instance, uv and $u^{\bullet}v^{\bullet}$, where $u \in C_i$, $v \in C_j$, $1 \le i, j \le k$ and at least one of i, j is k, for example $j = k$. Then:

$$\begin{aligned}
\varphi_k(u^{\bullet})\varphi_k(v^{\bullet}) &= \varphi_k(u^{\bullet})\varphi_k(v_2v_1) = \varphi_k(u^{\bullet})(\varphi_k(v_2)\varphi_k(v_1)) = [\boldsymbol{G} \in \mathcal{B}_c] \\
&= \varphi_k(u^{\bullet})(\varphi_k(v_1)\varphi_k(v_2)) = \varphi_k(u^{\bullet})\varphi_k(v_1v_2) \\
&= \varphi_k(u_2u_1)\varphi_k(v) \\
&= (\varphi_k(u_2)\varphi_k(u_1))\varphi_k(v) = [\boldsymbol{G} \in \mathcal{B}_c] \\
&= (\varphi_k(u_1)\varphi_k(u_2))\varphi_k(v) = \varphi_k(u)\varphi_k(v).
\end{aligned}$$

Thus, $\varphi_k(x)$ does not depend on the presentation of x. Hence, φ_{k+1} is a well defined mapping. Note that φ_k is a homomorphism for every $k \ge 1$.

Put $\varphi = \cup\{\varphi_k : k \ge 0\}$. Then φ is a mapping from H into G that is an extension of λ and it is a homomorphism from $\boldsymbol{H}$ into $\boldsymbol{G}$. Namely, if x is prime in $\boldsymbol{H}$, then $\varphi(x) = \varphi_0(x) = \lambda(x)$. Let $x \in H \setminus P$ and let (u, v) be the pair of divisors of x. Then, there are $i, j \ge 0$, such that $u \in C_i$, $v \in C_j$. If $m = \max\{i, j\}$, then $u, v \in P_m$. So, $\varphi(x) = \varphi(uv) = \varphi_{m+1}(uv) = \varphi_m(u)\varphi_m(v) = \varphi(u)\varphi(v)$. Hence, $\boldsymbol{H}$ has the universal mapping property for $\mathcal{B}_c$ over P. Since $\boldsymbol{H}$ is $\mathcal{B}_c$-injective, it follows that $\boldsymbol{H} \in \mathcal{B}_c$, and therefore $\boldsymbol{H}$ is $\mathcal{B}_c$-free groupoid over P. □

Remark The following question remains open: are there $\mathcal{B}_c$-injective groupoids that are not $\mathcal{B}_c$-free?

We state some notes on subgroupoids of $\mathcal{B}_c$-injective groupoids.

Let $\boldsymbol{Q}$ be a subgroupoid of a groupoid $\boldsymbol{G}$. If $\boldsymbol{G}$ is reversible, then there is a reversion $^{\triangleright} : Q \to Q$ defined as usual. Namely, let $a \in Q$. If a is not prime in Q, then $a = bc$ for some $b, c \in Q$ and we can put $a^{\triangleright} = a^{\bullet}$. If a is prime in $\boldsymbol{Q}$, put $a^{\triangleright} = a$. Thus, every subgroupoid of a reversible groupoid is reversible. Specially, $(R, *)$ is reversible. Note that the reversion on $\boldsymbol{Q}$ is not necessarily the restriction of the reversion on $\boldsymbol{G}$, since prime elements in $\boldsymbol{Q}$ are not necessarily prime elements in $\boldsymbol{G}$.

Proposition 3.10 *The class of $\mathcal{B}_c$-injective groupoids is hereditary.*

Proof. Let $\boldsymbol{H}$ be a $\mathcal{B}_c$-injective groupoid and let $\boldsymbol{Q}$ be a subgroupoid of $\boldsymbol{H}$. In order to show that $\boldsymbol{Q}$ is a $\mathcal{B}_c$-injective groupoid it suffices to check the conditions (2) and (3) of Def. 3.7.

Let z be a nonprime element in $\boldsymbol{Q}$. Then (2) directly follows from Prop.3.5. To show the "if part" of (3), let $z^{\triangleright} = z$. Then $z^{\triangleright} = z^{\bullet}$ in $\boldsymbol{H}$, where $^{\bullet}$ is the reversion on $\boldsymbol{H}$. Since $\boldsymbol{H}$ is a $\mathcal{B}_c$-injective groupoid, it follows that z is a square. For the "only if part", let z be a square in $\boldsymbol{Q}$. Then z is a square in $\boldsymbol{H}$ and thus $z^{\bullet} = z$. Since $z^{\bullet} = z^{\triangleright}$, it follows that $z^{\triangleright} = z$. □

There are subgroupoids of an infinite rank in a $\mathcal{B}_c$-free groupoid of rank 1.

Proposition 3.11 *Let $\boldsymbol{H} = (H, \cdot)$ be a $\mathcal{B}_c$-free groupoid with one element generating set $\{a\}$ and let $C = \{c_k : k \ge 1\}$ be defined by:*

$$c_1 = a^2, \qquad c_{k+1} = ac_k.$$

Then the rank of the subgroupoid $\boldsymbol{Q}$ of $\mathbf{H}$ is infinite.

Proof. It is easily seen that $c_i = c_j \Rightarrow i = j$ and thus C is infinite. Every element $c_k \in C$ is prime in $\boldsymbol{Q}$, since c_k, for $k \ge 1$, is not a product of elements in $\boldsymbol{Q}$. □

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