SEMIMARTINGALE THEORY AND COMPLETE MARKETS

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Abstract

We are interested in the application of the semimartingale theory in the complete market model consisting of two traded assets. Through the Girsanov theorem the unique martingale measure can be found and the replicating strategy can be constructed. This is discussed on the Black-Scholed market model.

Key words: Semimartingales, complete market, unique martingale measure, Girsanov theorem, Black-Scholes model

I. Introduction

In an arbitrage-free financial market model consisting of risky and riskless assets we would like to price a new derivative security in the market by a self-financing portfolio of the existing assets in the market. If this is possible for any integrable contingent claim the market is said to be complete. It has been shown in [1] that the completeness of the market is equivalent to the martingale representation property and to the existence of a unique equivalent martingale measure $\mathbb{Q}$. Then, a replicating strategy for the claim can be constructed using only this measure, which also gives the unique price of the claim.

Here, with an application of the Girsanov theorem it is obtained that the process *X* is a semimartingale under the physical measure $\mathbb{P}$. This is why it is justified in the first place to assume that the price process is modeled by a semimartingale.

This brings us to discussion of the application of the semimartingale theory, Girsanov theorem and the martingale property of the complete market model. We are interested in the characteristics of the model of the basic portfolio, together with the view that different investors have on it. We discuss the meaning of the unique martingale measure and see how it helps us in constructing the replicating strategy. After discussing the theoretical basics, we'll see the application on the Black-Scholes model, where the price process is driven by a Brownian motion. At the end we give an example of the simulated process under the physical and the equivalent martingale measure.

II. Complete market model

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a probability space with a filtration that satisfies the usual conditions. Here T denotes the finite time horizon, $\mathcal{F}_T = \mathcal{F}$ and the σ-algebra $\mathcal{F}_t$ represents the information observable at time t.

We observe a market consisting of 2 traded assets[1]. One is *riskless asset* (bond) whose price process $(B_t)_{t\in[0,T]}$ is assumed to be strictly positive. The other is *risky asset* (stock) and its change of price is modeled by a stochastic process $(X_t)_{t\in[0,T]}$[2] with continuous paths. The discounted process $\frac{X_t}{B_t}$ is of our main interest and therefore we assume that $B \equiv 1$. We keep the notation X_t for the discounted process. The discounted price process X_t is assumed to be a semimartingale under the physical measure $\mathbb{P}$, and this will be justified with a result from the next section. This way we can use the powerful semimartingale theory in our model. Therefore we introduce here the definition and a basic result for semimartingales, needed later in the discussion.

An adapted, càdlàg process *X* is a *semimartingale* if it can be written as $X_t = M_t + A_t$, $t \geq 0$ where *M* is a local martingale and *A* is a process with locally finite variation and $A_0 = 0$.

In general this decomposition is not unique, but when *X* is continuous, as in our model, it is unique, with *M* and *A* also continuos. Semimartingales are of great importance in the theory of stochastic integration - they are the most general, reasonable stochastic integrators. It is the largest class of integrators on which are preserved some convergence and approximation properties for the operation of stochastic integration. They also form a vector space invariant under stopping, localization, certain change of filtration, C^2-transformation (Itô's formula) and absolute continuous change of probability measure[3]. We discuss here this last property, given by the following theorem.

Theorem 1. *Let $X = M + A$ be a continuous $\mathbb{P}$-semimartingale. Let $\mathbb{Q}$ be an equivalent probability measure, with a Radon-Nikodym derivative $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$ and $Z_t = \mathbb{E}_{\mathbb{P}}(Z|\mathcal{F}_t)$ the coresponding martingale density* [4]*. Then X is also a continuous $\mathbb{Q}$-semimartingale with a decomposition $X = N + B$ where*

$$N_t = M_t - \int_0^t \frac{1}{Z_s} d\langle Z, M\rangle_s \qquad t \in [0,T] \tag{1}$$

is a $\mathbb{Q}$-local martingale and $B = X - N$ has locally finite variation.

An investor can create an investment portfolio whose dynamics is described by a trading strategy $\phi = (\alpha, \beta)$. α and β describe the amounts invested at time t in the stock and the bond respectively. By definition, α is a predictable process with $\alpha \in L^2(\mathbb{P}_X)$. The amount of the stock should be determined before observing the information at time t, i.e. before

[1] The model can be easily extended to a case with n stocks.

[2] Further in the discussion, for all the processes we take $t \in [0, T]$, unless otherwise specified.

[3] We refer to [2] for details on part of the theory presented here.

[4] *Z_t is defined here on [0,T] with $Z = Z_T$.*

knowing the next stock price change. This is the meaning of the predictability of the process α. β is in general an addapted process. This means that the amount in the bond can be fixed after time t, and by that adjust the value of the portfolio at some desired level.

The *value process* of the portfolio is given by

$$V_t = \alpha_t X_t + \beta_t \tag{2}$$

The value process is right-continuous with $V_t \in L^2(\mathbb{P})$. The *accumulated gain* and the *accumulated cost* of the strategy at time t are given by $\int_0^t \alpha_s dX_s$ and $C_t = V_t - \int_0^t \alpha_s dX_s$ respectively. We will be interested in *self-financing* strategies ($C_t = C_0$ for $t \in [0,T]\mathbb{P}$ a.s.). For such a strategy, the total wealth depends only on the initial investment V_0 and the gains and loses come only from the stock price changes, i.e. $V_t = V_0 + \int_0^t \alpha_s dX_s$.

A *contingent claim* (or a derivative security) is a financial contract whose value at the time of maturity T depends on the value of the underlying assets at time T. It is the payoff of some financial instrument at time T and is modeled as an $\mathcal{F}_T$-measurable random variable *H* (here with an assumption $H \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$). For a given contingent claim *H* we are looking for self-fainancing strategies ϕ s.t. $V_T = H$, $\mathbb{P}$ a.s. Such a strategy is said to be *H-admissible*, i.e. it replicates *H*. If such a strategy exists, *H* is refered to as a *attainable* contingent claim.

The main question is how, for a given contingent claim *H* can we find a trading strategy $\phi = (\alpha, \beta)$ that replicates *H*. This would mean that at time T holding the portfolio with the stock and the bond or holding the contingent claim would make no difference, since $\mathbb{P}$-almost surely their values would be the same. This can also help us find the fair price of the contract.

To give an answer to these questions we need the main result concerning the completeness of the market model.

III. Martingale property of the complete market

In [1] the Harrison and Pliska have shown the following theorem

Theorem 2. *The following statements are equivelent:*

a) *Every contingent claimis in the presented model is attainable,*
b) *Any martingale M can be represented as* $M_t = M_0 + \int_0^t \gamma_s dX_s$*, where* γ *is a predictable process integrable with respect to the semimartingale X,*
c) *There exists a unique measure* $\mathbb{Q}$ *equivalent to* $\mathbb{P}$ *under which the process X is a martingale.*

Market model in which a) holds is said to be *complete* under the physical measure $\mathbb{P}$. b) is known as the *Martingale representation property* (or $It\hat{o}$ representation). We refer to $\mathbb{Q}$ in c) as the equivalent martingale measure. For more detailed discussion we refer to [1] and [3].

According to the last theorem, there exists a unique probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the discounted process is a martingale under $\mathbb{Q}$. Let $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$. Since *X* is a continuous martingale under $\mathbb{Q}$, it is also a $\mathbb{Q}$-semimartingale with a representation $X_t = N_t + B_t$ where B=0 $\mathbb{Q}$ a.s. Girsanov theorem states that *X* is a semimartingale under $\mathbb{P}$, with a decomposition,

$$\mathrm{X}_t = \underbrace{X_t - \int_0^t Z_s d\left\langle Z^{-1}, X\right\rangle_s}_{M_t} + \underbrace{\int_0^t Z_s d\left\langle Z^{-1}, X\right\rangle_s}_{A_t}$$

This justifies our assumption that X is a semimartingale under the basic measure $\mathbb{P}$.

The measure $\mathbb{Q}$ will help us find the replicating strategy for an integrable contingent claim *H*. With $H_t = \mathbb{E}(H|\mathcal{F}_t)$ a martingale is defined, which, according to the previous theorem, has the $It\hat{o}$ representation

$$H_t = H_0 + \int_0^t \gamma_s dX_s \tag{3}$$

Let

$$\alpha_t := \gamma_t, \quad V_t := H_0 + \int_0^t \gamma_s dX_s, \tag{4}$$

If we put $\beta_t := V_t - \alpha_t X_t$ we get a strategy $\phi = (\alpha, \beta)$ with a constant cost $C_t = H_0$. Starting with an initial investment $V_0 = H_0$ and putting the amounts α and β in the stock and the bond, we will have $V_T = H$, i.e. we will replicate the claim *H*. Now the question is: How to determine γ?

Since *X* is a $\mathbb{Q}$-martingale, from (4) and the properties of the $It\hat{o}$-integral, *V* is also a $\mathbb{Q}$-martingale and

$$V_t = \mathbb{E}_{\mathbb{Q}}(V_T|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(H|\mathcal{F}_t)^5$$

For the covariation process $\langle V, X\rangle_t$ we have

$$\begin{aligned} \langle V, X\rangle_t &= \langle H_0 + \int_0^\cdot \gamma_s dX_s, X\rangle_t \\ &= \langle H_0, X\rangle_t + \langle \int_0^\cdot \gamma_s dX_s, X\rangle_t \\ &= \int_0^t \gamma_s d\langle X\rangle_s \end{aligned} \tag{5}$$

As a result $\gamma_t = \frac{d\langle V,X\rangle_t}{d\langle X\rangle_t}$ is the Radon-Nikodym derivative of the measures generated by the processes $\langle V, X\rangle$ and $\langle X\rangle$. By this we have constructed the replicating strategy $(\gamma, V - \gamma X)$ using only the contingent claim, the price process and the unique equivalent martingale measure $\mathbb{Q}$.

IV. Application to the Black-Scholes model

The Black-Scholes model describes a market consisting of a bond and a stock with price processes *B* and *X* respectively. The discounted process *X* is modeled by a Brownian motion *W* and the probability space we work on is $(\Omega, \mathcal{F}, (\mathcal{F}_t^W)_{t\in[0,T],\mathbb{P}})$. The dynamics of the process is given by the following SDE

[5] In general this process is defined by $V_t = \mathbb{E}_{\mathbb{Q}}(\frac{H}{B_T}|\mathcal{F}_t)$

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x_0 \tag{6}$$

The solution of this SDE is given by

$$X_t = X_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma dW_t} \tag{7}$$

where $\theta = \frac{\mu}{\sigma}$ is the *market price of risk*.

The process *X* is a semimartingale under the basic measure $\mathbb{P}$. We would like to find the unique equivalent martingale measure $\mathbb{Q}$ under which *X* becomes a martingale. Let $Z_t = e^{-\theta W_t - \frac{1}{2}\theta^2 t}$ on [0,T] be the solution of the SDE $dZ_t = -\theta Z_t dW_t$. It satisfies the Novikov condition, so it is a martingale density for some measure $\mathbb{Q}$ equivalent to $\mathbb{P}$[6]. Let us see how the semimartingale $W^* = W_t + \theta t$ changes under the measure $\mathbb{Q}$. The martingale part of its representation under $\mathbb{Q}$, by Girsanov theorem is

$$\begin{aligned} &W_t - \int_0^t \frac{1}{Z_s} d\langle Z, W\rangle_s = \\ &= W_t - \int_0^t \frac{1}{Z_s} d\langle \theta \int_0^s Z_u dW_u, W\rangle_u \\ &= W_t - \theta \int_0^t \frac{1}{Z_s} d(\int_0^s Z_u \langle W, W\rangle_u) \\ &= W_t - \theta \int_0^t \frac{1}{Z_s} d(\int_0^s Z_u du) \\ &= W_t + \theta \int_0^t \frac{1}{Z_s} Z_s ds = W_t + \theta t = W_t^* \end{aligned} \tag{8}$$

Therefore the process W_t^* is the Brownian motion under $\mathbb{Q}$. Then $X_t = X_0 e^{\frac{1}{2}\sigma^2 t + \sigma W_t^*}$ is the solution of the SDE $dX_t = \sigma X_t dW_t^*$. Therefore *X* is a martingale under $\mathbb{Q}$ and this is the unique equivalent martingale measure.

Now that we have $\mathbb{Q}$ and the Radon-Nikodym density we can proceed to find the optimal strategy as in the previous section. With this we have seen how the semimartingale theory and the Girsanov theorem help in answering the questions of change of measure and finding the replicating strategy for a contingent claim.

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[6] Details on martingale densities can be found in [4], [5] and [6].