Cooperation among Non-identical Oscillators Connected in Different Topologies

Conference Paper *in* Advances in Intelligent Systems and Computing · January 2015

DOI: 10.1007/978-3-319-09879-1_27

Miroslav Mirchev
Ss. Cyril and Methodius University in Skopje

Lasko Basnarkov
Ss. Cyril and Methodius University in Skopje

Ljupco Kocarev
Macedonian Academy of Sciences and Arts

Cooperation among non-identical oscillators connected in different topologies

Miroslav Mirchev[1], Lasko Basnarkov[1] and Ljupco Kocarev[1,2,3]

[1] Facuty of Computer Science and Engineering, Ss. Cyril and Methodius University, 1000 Skopje, Macedonia
{miroslav.mirchev,lasko.basnarkov}@finki.ukim.mk
[2] Macedonian Academy of Sciences and Arts, 1000 Skopje, Macedonia;
[3] BioCircuits Institute, University of California, San Diego, La Jolla, CA 92093 USA
lkocarev@ucsd.edu

Abstract. Various forms of oscillatory networks exist in our surrounding from neural cells to laser arrays. In many of these networks the nodes can go through a transient process of interaction and start oscillating in synchrony. Each of these nodes is characterized by its internal dynamics and changes its state accordingly. Using several forms of interactions, we numerically examine how the network dynamics is affected by network topology and potential random disturbances.

Keywords: Complex networks. Oscillators. Synchronization

1 Introduction

Our environment is full of various networks of entities exhibiting periodic oscillations, like interconnected neural and heart cells in biology, wireless sensor networks, laser arrays and electronic oscillators in engineering, and so on. The nodes in these networks interact and after some time can agree to oscillate synchronously as studied in [1] and [2]. The synchronization could be desirable as in sensor networks and laser arrays, or undesirable as in epilepsy seizures in the brain. In all these networks, the nodes are characterized by states and if the network interactions result in all the nodes reaching the same state, it is said that the network is completely synchronized. On the other hand, in case of networks with weak interactions the nodes can agree on an equal frequency of oscillation without exact match of their individual states (amplitude values). This type of networks have been widely studied in the literature and many models for representation are known with the most famous example being the Kuramoto model [3] given as

$$\dot{\theta}_i = \omega_i + \frac{\gamma}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \tag{1}$$

where θ_i is the phase of oscillation of each node of a population of N nodes, ω_i is the node's internal oscillating frequency, with which it would oscillate if isolated, and γ is a general coupling strength typically larger then zero. Despite the differences in the internal frequencies a synchronous mode of oscillation is possible in this kind of networks.

In a previous paper [4] we have studied cooperation in networks of non-identical oscillators, particularly the case of non-identical interactions. In that paper convergence criteria toward frequency synchronization were provided for a specific type of nonnegative and symmetrical interactions. Moreover, the behavior of other types of systems were also examined, like asymmetric connections, external fields and frustration due to random disturbances. In this paper we further study these type of networks, particularly focusing on the effects of random disturbances with different forms of coupling functions and the effects of network topology on the dynamical behavior. All these issues have been part of a wider study of complex networks with imperfections in [5].

The paper continues with Section 2 where we introduce the notation of networks of non-identical oscillators. We numerically study the phenomenon of frequency synchronization, focusing in Section 3 on the effects of random disturbances and examining in Section 4 the network topology effects, while Section 5 provides some conclusions.

2 Networks of non-identical oscillating nodes

We consider networks composed of N oscillating nodes whose dynamics can be represented by

$$\dot{x}_i = \omega_i + \gamma \sum_{j=1}^{N} a_{ij} f_{ij}(x_j - x_i), \tag{2}$$

where as in the Kuramoto model the phases lay on a unit circle S^1 ($x_i \in S^1$) and $x_i \in [0, 2\pi)$, the natural frequencies are $\omega_i \in \mathbb{R}$ and γ is a general coupling strength, while $\mathbf{A} = [a_{ij}]$, ($a_{ij} \geq 0$, $a_{ii} = 0$, $a_{ij} = a_{ji}$, $\forall i, j$) is an adjacency matrix representing the network's topology. The coupling functions f_{ij} are taken to be 2π-periodic and $f_{ij}(0) = 0$, $\forall i, j$.

One measure of network coherence is how close are the phases and to evaluate and visualize this type of network coherence an order parameter [3] can be used

$$re^{\mathrm{i}\Psi} = \frac{1}{N} \sum_{j=1}^{N} e^{\mathrm{i}x_j}, \tag{3}$$

where larger values mean larger coherence ($r \in [0, 1]$), while Ψ is the average phase of the population.

Another type of coherence is the network's synchrony and we use the following error function to evaluate it

$$e_{\Omega}(t) = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (\dot{x}_i(t) - \Omega)^2}, \tag{4}$$

which indicates how the oscillators' velocities are approaching the mean natural frequency $\Omega = (1/N)\sum_i \omega_i$. Sometimes the analysis require $< e_\Omega >$, the time average of $e_\Omega(t)$, which is calculated with excluded transient dynamics.

In our analysis we consider networks with random uniformly distributed initial phases and random natural frequencies following a triangular distribution in the range $[\omega_{min}, \omega_{max}] = [-0.5, 0.5]$ with a probabilistic density function's peak at $\omega_0 = 0$.

Typically the coupling functions are taken to be sinusoidal, as in the Kuramoto model [3]

$$f_{ij}(x_j - x_i) = \sin(x_j - x_i). \tag{5}$$

However, other types of coupling functions should also be analyzed as in reality the interactions are not exactly sinusoidal [6]. One simple case are linear coupling functions that are periodically repeated in the following way

$$f_{ij}(x_j - x_i) = (x_j - x_i - 2\pi k),\ \text{for } -\pi + 2k\pi < x_j - x_i < \pi + 2k\pi, \tag{6}$$

for $k = 0, \pm 1, \pm 2, \ldots$. Another case that we study are periodically repeated cubic coupling function of the form

$$f_{ij}(x_j - x_i) = (x_j - x_i - 2\pi k)^3,\ \text{for } -\pi + 2k\pi < x_j - x_i < \pi + 2k\pi, \tag{7}$$

for $k = 0, \pm 1, \pm 2, \ldots$.

In our study we numerically simulate networks consisting of $N = 100$ oscillating nodes. In Section 3 we use fully connected networks where $a_{ij} = 1, \forall i, j, i \neq j$ and $a_{ii} = 0, \forall i$, while in Section 4 we examine the effects of the network topology so not all adjacency elements have a value of one. The numerical integration of the equations of motion of the oscillators is performed using a fourth-order Runge-Kutta method with a fixed step $\Delta t = 0.001$.

3 Random disturbances effects

In reality the interactions among the nodes are prone to some environmental or internal random disturbances also called frustrations. In our paper [4] it was analytically and numerically shown that besides these frustrations, sinusoidally coupled oscillators can eventually agree on a common oscillation frequency, as previously observed in [7].

The random disturbances can be introduced by including elements $\phi_{ij} \in (-\pi/2, \pi/2)$, $\phi_{ij} \in \mathbb{R}$, $\forall i, j$ [8], thus the nodes dynamics takes the form

$$\dot{x}_i = \omega_i + \gamma \sum_{j=1}^{N} a_{ij} f_{ij}(x_j - x_i + \phi_{ij}). \tag{8}$$

As discussed in [4], if $\phi_{ji} = -\phi_{ij}$ and $\phi_{ii} = 0$ for all interactions, the oscillators can achieve frequency synchronization as the synchronized state is stable.

In this section we provide numerical results of the dynamics of systems of the form defined by (8) and their idealistic counter pairs as given by (2). The network is taken to be fully connected in order to separate the disturbances effects from the topology effects that are considered in the following section. The general coupling strength is chosen to be $\gamma = 0.01$, which is big enough to allow network synchronization but not too large.

The first four sub-figures of Fig. 1 visualize the time evolution of the oscillators' phases in several different coupled networks. A polar coordinate system is used where the time flow is shown on the radial coordinate and the phases are on the angular coordinate. Thus, from the center of the circle toward the periphery the time t increases from 0 to 200 and the phase evolution of each oscillator is represented with a single continuous line. Fig. 1a shows the phase evolution in a linearly coupled network without frustration, where although the phases are not exactly matched ($r = 0.9811$) synchronization is achieved and the phases evolve at an equal rate. On the other hand, in Fig. 1b the linear interactions in the network are frustrated, which introduces a larger phase dispersion ($r = 0.9298$), though still allowing frequency synchronization among the oscillators. We also consider cubic coupling functions, first without frustration in Fig. 1c. This cubic coupling introduces clustered syncrhony that makes the order parameter low $r = 0.122$, as also observed previously in [4]. The introduction of the random disturbances in the cubic coupling in Fig. 1d prevents the network clustering and increases the coherence ($r = 0.9318$), while still allowing the oscillators to rotate their phases at an equal rate.

In Fig. 1e is shown how the synchronization error e_{Ω} reduces for the different types of coupling functions. As expected, the error reduces more rapidly with cubic coupling than with linear coupling, while the random disturbances though still allowing synchronization significantly slow down the convergence rate. The evolution of the order parameter r is shown in Fig. 1f and similar conclusions can be drawn as from the previous sub-figures. With linear coupling the frustration reduces the coherence, while with cubic coupling the coherence is increased in the presence of random disturbances due to the avoided clustering. The convergence rate of the order parameter is not as influenced as the synchronization error.

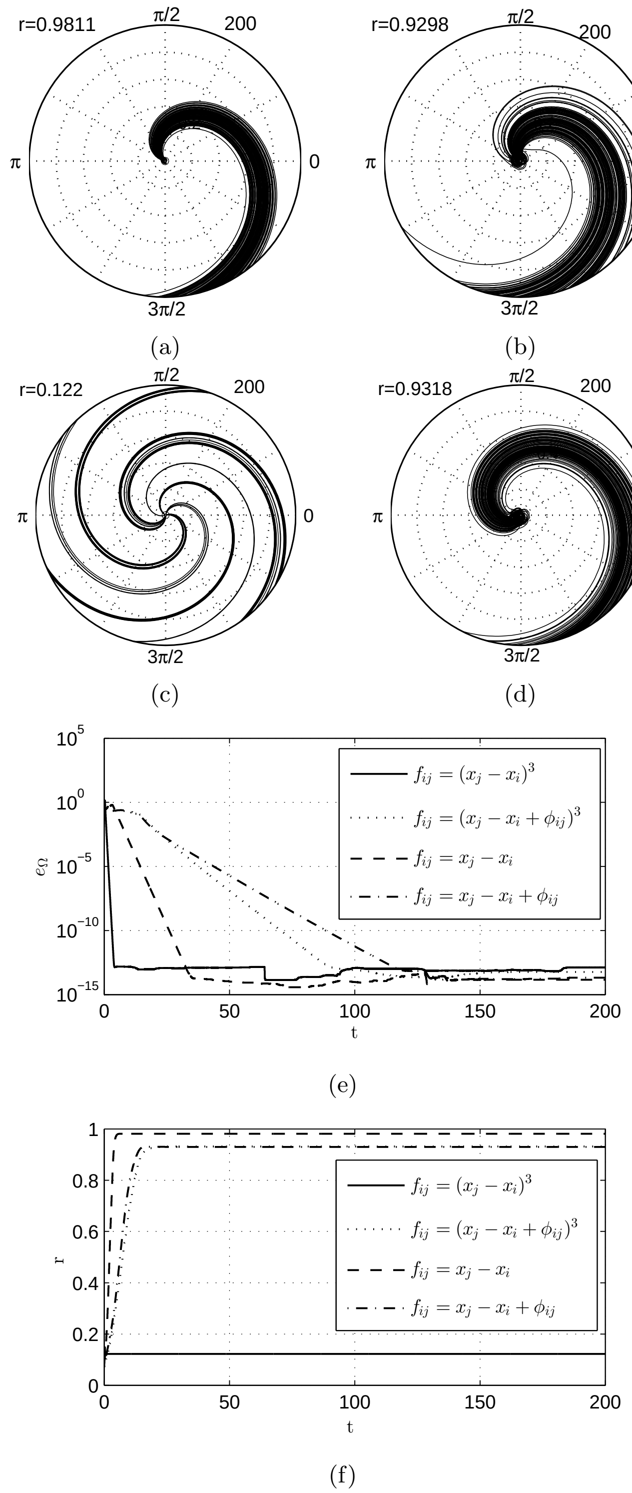

Fig. 1: Phase evolution in $t \in [0, 200]$ in a fully connected network with $\gamma = 0.01$ and (a) $f_{ij} = (x_j - x_i)$; (b) frustrated $f_{ij} = ((x_j - x_i) + \phi_{ij})$; (c) $f_{ij} = (x_j - x_i)^3$ and (d) frustrated $f_{ij} = ((x_j - x_i) + \phi_{ij})^3$; (e) synchronization error e_Ω and (f) coherence r.

4 Topological effects

In the networks studied in the previous section the oscillators were fully connected, however, in reality network interactions follow certain patterns that form the network topology. In this section, we study network dynamics in random networks generated using the Erdös-Rényi (ER) model [9] and scale-free networks generated using the Barabási-Albert (BA) model [10]. In all cases a care should be taken that the generated network is connected, i.e. there is a path among all node pairs, as otherwise synchronization is not achievable. Some analyses of the topology effects on the synchronization properties in a Kuramoto model with scale-free topology with standard sinusoidal couplings were done in [11, 12], and [13], using numerical simulations and different analytical approaches. However, a consensus have not been reached for the critical coupling gain at which synchronization occurs as also noted in [14]. In [15] and [16] both scale-free (BA) networks and random (ER) networks are examined in which the oscillator's natural frequencies are correlated to their degree of connectivity. In our study we do not assume any correlation among the degrees of the nodes and their internal frequencies.

The model for random network generation developed by Erdös and Rényi (ER) creates a graph $G(N, p)$ consisting of N nodes, where each of the possible links among the nodes exist with a probability p. For lower values of p this model can produce disconnected network parts, hence, if we need a connected network the connectivity should be checked and if the network is not connected the whole procedure could be repeated. Here we generate networks of $N = 100$ oscillators with link probability $p = 0.1$, which results in a network of about 500 links.

The Barabási-Albert model can be used for creating scale-free networks. The model requires an initial seed network to which gradually new nodes are added with L_N connections per node. This procedure, also known as preferential attachment, resembles a well known phenomenon where "the rich get richer", present for example in genetic networks, the World Wide Web, the Internet, social networks, etc. To assure that the networks with different topologies are comparable, the number of nodes and links is kept the same, so gradually to the seed network new nodes with $L_N = 5$ connections are added until we reach $N = 100$ nodes and 500 links.

Example networks with these topologies are given in Fig. 2. The comparison of the results in Fig. 3 show that in this case ER networks synchronize more easily than BA networks, because for both linear and sinusoidal coupling synchronization occur for lower values of γ. The coherence r is similar for both types of networks, and with the increase of the coupling strength r rises just slightly quicker in ER networks. It can be noticed by looking at the coherence r that for linear coupling clustering occurred in the random network for some values of γ, while in the scale-free network a step-wise increase was observed at some points due to the hierarchical structure. Similar results were reported in [16] for sinusoidally coupled networks in which node's degrees and natural frequencies are correlated.

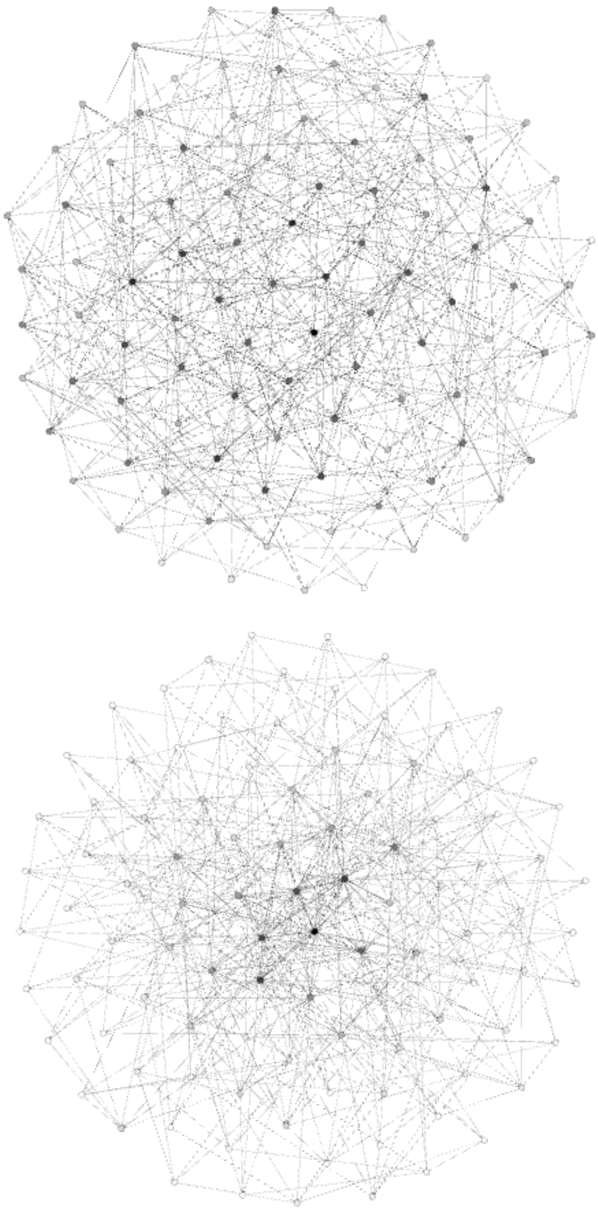

Fig. 2: Two example network topologies where the nodes with higher degree are colored darker and placed more centrally: (top) random network – Erdös and Rényi and (bottom) scale-free network – Barabási-Albert. Visualization is done in Gephi using the Fruchterman–Reingold algorithm.

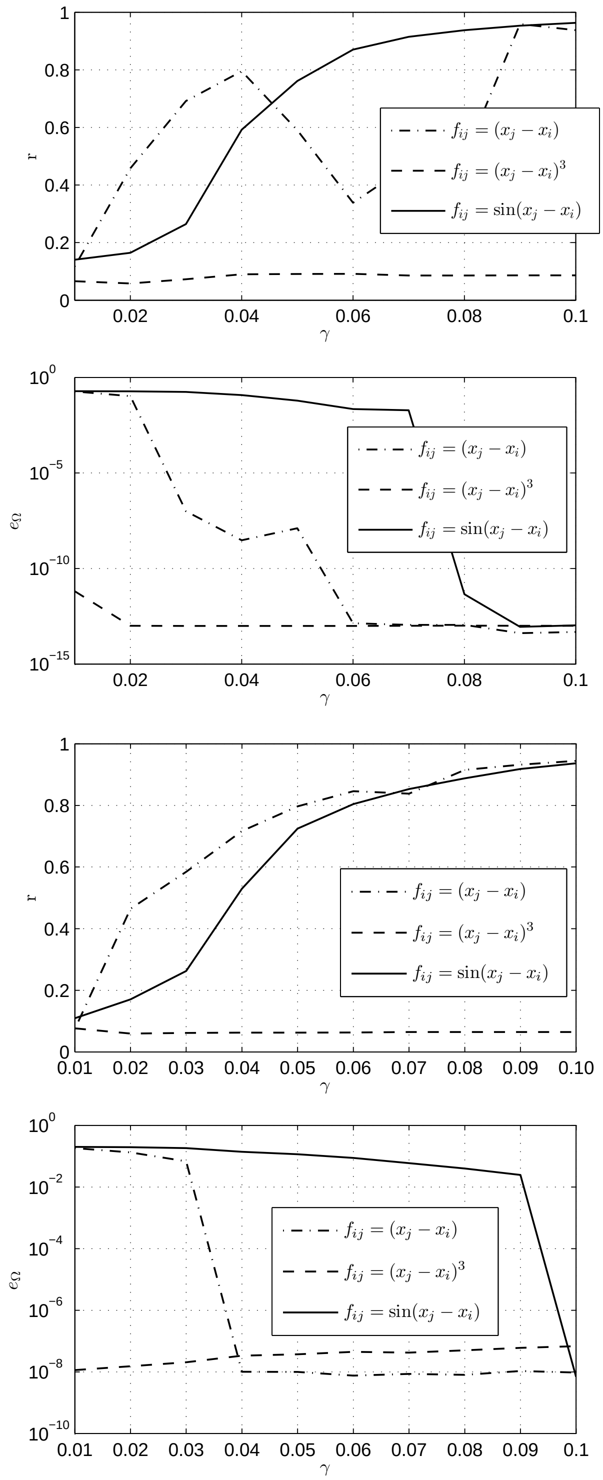

Fig. 3: The coherence r and the synchronization error e_Ω in networks of $N = 100$ nodes and 500 links generated using: (top two) the ER model, and (bottom two) the BA model for different coupling functions.

5 Conclusion

This paper provides additional examination of the process of cooperation in networks of non-identical oscillators through several forms of interaction. Particularly, the focus is on the effects of possible random disturbances and the network topology on the network dynamics. Possible achievement of frequency synchronization was confirmed for linear and cubic coupling functions, in addition to the known results with the standard sinusoidal function. The study of the topology effects showed that frequency synchronization can be achieved more easily in random networks than in scale-free networks, although, typically complete synchronization happens more easily in scale-free networks.

Acknowledgements. The authors thank the Faculty of computer science and engineering at the Ss. Cyril and Methodius University in Skopje, under the DYSENE (Dynamical sensor networks) project for financial support.

References

1. Wiener, N.: Nonlinear Problems in Random Theory. MIT Press, Cambridge, MA, USA (1958)
2. Winfree, A.T.: Biological rhythms and the behavior of populations of coupled oscillators. J. Theor. Biol. **16**(1) (1967) 15–42
3. Kuramoto, Y.: Chemical Oscillations, Waves, and Turbulence. Springer, Berlin, Germany (1984)
4. Mirchev, M., Basnarkov, L., Corinto, F., Kocarev, L.: Cooperative phenomena in networks of oscillators with non-identical interactions and dynamics. IEEE Trans. Circuits Syst. I, Reg. Papers **61**(3) (2014) 811–819
5. Mirchev, M.: Cooperative processes in complex networks with imperfections. PhD thesis, Politecnico di Torino, Italy (2014)
6. Daido, H.: Order function and macroscopic mutual entrainment in uniformly coupled limit-cycle oscillators. Prog. Theor. Phys. **88**(6) (1992) 1213–1218
7. Daido, H.: Quasientrainment and slow relaxation in a population of oscillators with random and frustrated interactions. Phys. Rev. Lett. **68**(7) (1992) 1073–1076
8. Sakaguchi, H., Kuramoto, Y.: A soluble active rotator model showing phase transitions via mutual entrainment. Prog. Theor. Phys. **76**(3) (1986) 576–581
9. Erdös, P., Rényi, A.: On the evolution of random graphs. Publ. Math. Inst. Hungar. Acad. Sci. **5** (1960) 17–61
10. Barabási, A.-L., Albert, R: Emergence of scaling in random networks. Science **286**(5439) (1999) 509–512
11. Moreno, Y., Pacheco, A.F.: Synchronization of kuramoto oscillators in scale-free networks. Europhys. Lett. **68**(4) (2004) 603–609
12. Ichonmiya, T.: Frequency synchronization in a random oscillator network. Phys. Rev. E **70**(2) (2004) 026116
13. Restrepo, J.G., Ott, E., Hunt, B.R.: Onset of synchronization in large networks of coupled oscillators. Phys. Rev. E **71**(3) (2005) 036151
14. Boccaletti, S., Latora, V., Moreno, Y., Chavez, M., Hwang, D.-U.: Complex networks: Structure and dynamics. Phys. Rep. **424**(4-5) (2006) 175–308

15. Gómez-Gardeñes, J., Gómez, S., Arenas, A., Moreno, Y.: Explosive synchronization transitions in scale-free networks. Phys. Rev. Lett. **106**(12) (2011) 128701
16. Coutinho, B.C., Goltsev, A.V., Dorogovtsev, S.N., Mendes, J.F.F.: Kuramoto model with frequency-degree correlations on complex networks. Phys. Rev. E **8**(3) (2013) 032106