FREE OBJECTS IN THE VARIETY OF GROUPOIDS DEFINED BY THE IDENTITY $xx^{(m)} \approx x^{(m+1)}$

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ABSTRACT. A construction of free objects in the variety $\mathcal{V}_{(m)}$ of groupoids defined by the identity $xx^{(m)} \approx x^{(m+1)}$, where m is a fixed positive integer, and (k) is a transformation of a groupoid $\boldsymbol{G} = (G, \cdot)$ defined by $x^{(0)} = x$, $x^{(k+1)} = (x^{(k)})^2$, is given. A class of injective groupoids in $\mathcal{V}_{(m)}$ is defined and a corresponding Bruck theorem for this variety is proved. It is shown that the class of free groupoids in $\mathcal{V}_{(m)}$ is a proper subclass of the class of injective groupoids in $\mathcal{V}_{(m)}$.

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1. Preliminaries

Let $\boldsymbol{G} = (G, \cdot)$ be a groupoid, i.e. an algebra with one binary operation.

For any nonnegative integer k we define a transformation $(k) : x \to x^{(k)}$ of G as follows:

$$x^{(0)} = x, \qquad x^{(k+1)} = (x^{(k)})^2.$$

(Here, x^k is defined by: $x^1 = x$, $x^{k+1} = x^k x$; ex: $x^4 = ((xx)x)x$.)

We say that $x^{(k)}$ is the k *square of* x and $x^{(1)} = x^2$ the *square* of x.

By induction on p and q one can show ([**2**]) that in any groupoid $\boldsymbol{G} = (G, \cdot)$ $(x^{(p)})^{(q)} = x^{(p+q)}$ for any $x \in G$ and any nonnegative integers p, q.

The variety of groupoids defined by the identity $xx^{(m)} \approx x^{(m+1)}$ will be denoted by $\mathcal{V}_{(m)}$, for a fixed positive integer m. (The variety $\mathcal{V}_{(1)}$ is investigated in [**3**].)

If $\boldsymbol{G} \in \mathcal{V}_{(m)}$, then by induction on p, one obtains:

$$(\forall x \in G, p \geq 0)\ \ x^{(p)} x^{(p+m)} = x^{(p+m+1)}.$$

An element $a \in G$ is said to be *2-primitive* in $\boldsymbol{G}$ if and only if

$$(\forall x \in G, p \geq 0)\ \ (a = x^{(p)}\ \Rightarrow\ p = 0).$$

In the sequel B will be an arbitrary nonempty set whose elements are called variables. By T_B we will denote the set of all groupoid terms over B in the signature $\cdot$. The terms are denoted by $t, u, v, \ldots, x, y, \ldots$ $\boldsymbol{T}_B = (T_B, \cdot)$ is the absolutely free groupoid with the free basis B, where the operation is defined by $(u, v) \mapsto uv$. It is well known (Bruck theorem for $\boldsymbol{T}_B$, [**1**]) that the following two properties characterize $\boldsymbol{T}_B$:
(*i*) $\boldsymbol{T}_B$ is *injective*, i.e. the operation $\cdot : (u, v) \mapsto uv$ is an injection.
(*ii*) The set B of primes in $\boldsymbol{T}_B$ is nonempty and generates $\boldsymbol{T}_B$.
(An element a in a groupoid $\boldsymbol{G} = (G, \cdot)$ is said to be *prime* in $\boldsymbol{G}$ if and only if $a \neq xy$, for any $x, y \in G$.)

For any term v of T_B we define the *length* $|\, v\,|$ of v and the *set of subterms* $P(v)$ of v in the following way:

$$|\, b\,| = 1,\ |\, tu\,| = |\, t\,| + |\, u\,|;\ P(b) = \{b\},\ P(tu) = \{tu\} \cup P(t) \cup P(u),$$

for any variable b and any terms t, u of T_B.

Bellow we consider a few properties of $x^{(k)}$ in $\boldsymbol{T}_B$ that can be shown by induction on p.

Proposition 1.1.. *If $t, u \in T_B$ and p, q are nonnegative integers, then:*
a) $|\, t^{(p)}\,| = 2^p |\, t\,|$.
b) $t^{(p)} = u^{(p+q)}\ \Rightarrow\ t = u^{(q)}$.
c) If t, u are 2-primitive elements in T_B, then: $t^{(p)} = u^{(q)}\ \Leftrightarrow\ t = u,\ p = q$.

Proof. c) We assume that $p \leq q$, i.e. $q = p + k$, for any $k \geq 0$. Then from $t^{(p)} = u^{(q)}$ we have that $t^{(p)} = u^{(p+k)}$. By b) it follows that $t = u^{(k)}$. However, t is 2-primitive in $\boldsymbol{T}_B$, therefore $k = 0$. Thus $t = u^{(0)} = u$, and $p = q$. The converse is obvious. □

By Prop.1.1. c) it follows directly that:

Proposition 1.2.. *For any $t \in T_B$, there is a unique 2-primitive term $\alpha \in T_B$ and a unique nonnegative integer p, such that $t = \alpha^{(p)}$.* □

We say that α is the 2-*base* of t and p is the 2-*exponent* of t; we denote them by $\underline{t} = \alpha$, $[t] = p$, respectively.

2. A construction of free objects in $\mathcal{V}_{(m)}$

Assuming that B is a nonempty set and $\boldsymbol{T}_B = (T_B, \cdot)$ the absolutely free groupoid with the free basis B, we are looking for a *canonical groupoid* ([**4**]) in $\mathcal{V}_{(m)}$, i.e. a groupoid $\boldsymbol{R} = (R, *)$ with the following properties:
i) $B \subset R \subset T_B$; *ii*) $tu \in R \Rightarrow t, u \in R$; *iii*) $tu \in R \Rightarrow t * u = tu$
iv) $\boldsymbol{R}$ is a free groupoid in $\mathcal{V}_{(m)}$ with the free basis B.
Define the carrier R of the desired groupoid $\boldsymbol{R}$ by:

$$R = \{t \in T_B : (\forall x \in T_B)\ xx^{(m)} \notin P(t)\}. \tag{2.1}$$

The following properties of R are obvious corollaries of (2.1).

Proposition 2.1.. a) R satisfies *i*) and *ii*).

b) $t, u \in R \Rightarrow \{tu \notin R \Leftrightarrow u = t^{(m)}\}$.

c) $t, u \in T_B \Rightarrow \{tu \in R \Leftrightarrow t, u \in R\ \&\ u \neq t^{(m)}\}$.

d) $t \in R,\ p \geq 1 \Rightarrow t^p \in R,\ t^{(p)} \in R$. □

We define an operation $*$ on R as follows.

$$t, u \in R \Rightarrow t * u = \begin{cases} tu, & \text{if } tu \in R \\ t^{(m+1)}, & \text{if } u = t^{(m)}. \end{cases} \tag{2.2}$$

From (2.2) and Prop.2.1. d), by induction on p, we obtain:
e) $t \in R,\ p \geq 1 \Rightarrow t_*^p = t^p,\ t_*^{(p)} = t^{(p)}$,
where t_*^k is defined by: $t_* = t$, $t_*^{k+1} = t_*^k * t$ and $t_*^{(p)}$ is the p square of t in $\boldsymbol{R}$.

By a direct verification one can show that the operation $*$ is well-defined, i.e. $\boldsymbol{R} = (R, *)$ is a groupoid. From (2.2) it follows that if $tu \in R$, then $t, u \in R\ \&\ t * u = tu$ (i.e. $\boldsymbol{R}$ satisfies *ii*) and *iii*)). By the property e) and (2.2), we obtain that $t * t_*^{(m)} = t * t^{(m)} = t^{(m+1)} = t_*^{(m+1)}$, i.e. $\boldsymbol{R} \in \mathcal{V}_{(m)}$.

The set of primes in $\boldsymbol{R}$ coincides with B and generates $\boldsymbol{R}$. Namely, every $b \in B$ is prime in $\boldsymbol{R}$, since $b \neq t * u$, for any $t, u \in R$. To show that no element of $R \setminus B$ is prime in $\boldsymbol{R}$, let $t \in T_B \setminus B$ be a term belonging to R. Then there are $t_1, t_2 \in T_B$, such that $t = t_1 t_2$. By the fact that $t \in R$, i.e. $t_1 t_2 \in R$, it follows that $t_1, t_2 \in R$ and $t = t_1 t_2 = t_1 * t_2$, i.e. t is not prime in $\boldsymbol{R}$. Let $\boldsymbol{Q}$ be the subgroupoid of $\boldsymbol{R}$ generated by B, $\boldsymbol{Q} = \langle B \rangle_*$. We will show that $R = Q$. Clearly, $Q \subseteq R$. To show that $R \subseteq Q$, let $t \in R$. If $t \in B$, then $t \in \langle B \rangle_* = Q$, i.e. $(t \in R \ \& \ |t| = 1 \Rightarrow t \in Q)$. Suppose that $(t \in R \ \& \ |t| \leq k \Rightarrow t \in Q)$ is true. If $t \in R$ is such that $|t| = k+1$, then $t = t_1 t_2$ in $\boldsymbol{T}_B$ and $|t_1|, |t_2| \leq k$. By the inductive hypothesis we have $t_1, t_2 \in Q$, and since $\boldsymbol{Q}$ is a groupoid, it follows that $t = t_1 t_2 = t_1 * t_2 \in Q$. Thus, $R \subseteq Q$. Therefore, $\boldsymbol{R} = \boldsymbol{Q} = \langle B \rangle_*$.

$\boldsymbol{R}$ has the universal mapping property ([**5**]) for $\mathcal{V}_{(m)}$ over B. Namely, let $\boldsymbol{G} \in \mathcal{V}_{(m)}$, $\lambda : B \to G$ be any mapping and $\varphi : T_B \to G$ be the homomorphism from $\boldsymbol{T}_B$ into $\boldsymbol{G}$ that extends λ. Let $t, u \in R$. If $tu \in R$, then $\varphi(t * u) = \varphi(tu) = \varphi(t)\varphi(u)$. If $tu \notin R$, then $u = t^{(m)}$ and $t * u = t^{(m+1)}$. Using the fact that $\varphi(t^{(p)}) = (\varphi(t))^{(p)}$ (it can be shown by induction on p), we obtain that $\varphi(t * u) = \varphi(t^{(m+1)}) = \varphi(t^{(m)} t^{(m)}) = \varphi(t^{(m)})\varphi(t^{(m)}) = (\varphi(t))^{(m)}(\varphi(t))^{(m)} = (\varphi(t))^{(m+1)} = \varphi(t^{(m+1)}) = [\boldsymbol{G} \in \mathcal{V}_{(m)}] = \varphi(t t^{(m)}) = \varphi(tu) = \varphi(t)\varphi(u)$.

Thus, $\varphi|_R : R \to G$ is a homomorphism that extends λ.

Therefore, the conditions *i)* - *iv)* at the begining of this section are fulfilled and thus we proved the following

THEOREM 2.1.. *The groupoid* $\boldsymbol{R} = (R, *)$, *defined by* (2.1) *and* (2.2), *is a canonical groupoid in* $\mathcal{V}_{(m)}$ *with a free basis* B. $\square$

As a consequence of the property e) and the definition of 2-primitive element we obtain the following

PROPOSITION 2.2.. *For any* $u \in R$, *there are a unique 2-primitive element* $t \in R$ *and a unique positive integer* p, *such that* $u = t_*^{(p)} = t^{(p)}$. $\square$

By a direct verification one can show that $(R, *)$ is a left cancellative groupoid. $(R, *)$ is not a right cancellative gropoid (ex: $t^{(1)} * t^{(m+1)} = t^{(m+2)} = t^{(m+1)} * t^{(m+1)}$; however $t^{(1)} \neq t^{(m+1)}$).

The following proposition will be used in the next section.

PROPOSITION 2.3.. *Let* $x \in R \setminus B$.
a) If x *is a 2-primitive element in* $\boldsymbol{R}$ *or* $x = \alpha^{(p)}$, *where* $[\alpha] = 0$, $1 \leq p \leq m$, *then there is a unique pair* $(u, v) \in R \times R$, *such that* $x = u * v$. (In that case

$x = uv$ and $v \neq u^{(m)}$.)

We say that (u, v) is *the pair of divisors* of x in $\boldsymbol{R}$.

b) If $x = t^{(m+p+1)}$*,* $p \geq 0$*, then* $x = t^{(p+m)} * t^{(p+m)} = t^{(p)} * t^{(p+m)}$*.*

Thus $(t^{(p+m)}, t^{(p+m)})$ and $(t^{(p)}, t^{(p+m)})$ are pairs of divisors of x. □

3. Injective objects in $\mathcal{V}_{(m)}$

In this Section we will give a characterization of the free groupoids in $\mathcal{V}_{(m)}$ by a wider class, called the class of injective groupoids ([**4**]) in $\mathcal{V}_{(m)}$. For that purpose, we use the properties of the corresponding canonical groupoid $(R, *)$ in $\mathcal{V}_{(m)}$ previously constructed, that concern the non-prime elements in $\boldsymbol{R}$, i.e. elements of $R \setminus B$.

We say that a groupoid $\boldsymbol{H} = (H, \cdot)$ is *injective* in $\mathcal{V}_{(m)}$ (i.e. $\mathcal{V}_{(m)}$-*injective*) if and only if the following conditions are satisfied:

(0) $\boldsymbol{H} \in \mathcal{V}_{(m)}$

(1) For any $a \in H$, there is a unique 2-primitive element $c \in H$ and a unique nonnegative integer k, such that $a = c^{(k)}$.

(We say that c is the 2-*base* of a and $k = [a]$ is the 2-*exponent* of a.)

(2) If $a \in H$ is a non-prime 2-primitive element in $\boldsymbol{H}$, then there is a unique pair $(c, d) \in H \times H$ such that $a = cd$ and $(\underline{c} \neq \underline{d} \ \vee \ (\underline{d} = \underline{c} \ \& \ [d] \neq m))$.

(In that case we say that (c, d) is the *pair of divisors* of a (we write $(c, d)|a$).)

(3) If $a \in H$ is such that $a = c^{(1)}$, $[c] = p$, $p \leq m - 1$, then (c, c) is the pair of divisors of a.

(4) $a^{(p+m+1)} = cd \ \& \ p \geq 0 \ \Leftrightarrow [c = d = a^{(p+m)} \ \vee \ (c = a^{(p)} \ \& \ d = a^{(p+m)})]$.

From the definition of $\mathcal{V}_{(m)}$-injective groupoid and Prop.2.3. we obtain that

Proposition 3.1.. *The class of free groupoids in* $\mathcal{V}_{(m)}$ *is a subclass of the class of* $\mathcal{V}_{(m)}$*-injective groupoids.* □

Theorem 3.1. (Bruck Theorem for $\mathcal{V}_{(m)}$). *A groupoid* $\boldsymbol{H}$ *is free in* $\mathcal{V}_{(m)}$ *if and only if* $\boldsymbol{H}$ *satisfies the following two conditions:*

(i) $\boldsymbol{H}$ *is* $\mathcal{V}_{(m)}$*-injective*

(ii) The set P *of primes in* $\boldsymbol{H}$ *is nonempty and generates* $\boldsymbol{H}$*.*

Proof. If $\boldsymbol{H}$ is free in $\mathcal{V}_{(m)}$ with a free basis B, then by Prop.3.1., $\boldsymbol{H}$ is $\mathcal{V}_{(m)}$-injective, and by the proof of Theorem 2.1., B is the set of primes in $\boldsymbol{H}$ and generates $\boldsymbol{H}$.

For the converse, it suffices to show that $\boldsymbol{H}$ has the universal mapping property for $\mathcal{V}_{(m)}$ over P. Therefore, define an infinite sequence of subsets $C_0, C_1, \dots$ of H by:

$$C_0 = P,\ C_1 = C_0C_0 = PP,$$
$$C_{k+1} = \{t \in H \setminus P :\ (c,d)|t\ \Rightarrow \{c,d\} \subseteq C_0 \cup C_1 \cup \dots \cup C_k\ \ \&\ \ \{c,d\} \cap C_k \neq \varnothing\}.$$

Then the following statements are true ([**4**]):

1) $(\forall k \geq 0)\ C_k \neq \varnothing$; 2) $a \in C_k \Rightarrow (\forall p \in \mathbb{N})\ a^{(p)} \in C_{k+p},\ \ k \geq 0$.
3) $p \neq q \Rightarrow C_p \cap C_q = \varnothing$; 4) $H = \bigcup\{C_k :\ k \geq 0\}$.

Let $\boldsymbol{G} \in \mathcal{V}_{(m)}$ and $\lambda : P \to G$ be a mapping. For any nonnegative integer k define a mapping $\varphi_k : C_k \to G$ by $\varphi_0 = \lambda$, and let φ_i be defined for each $i \leq k$. Let $a \in C_{k+1}$ and $(c,d)|a$ are such that $c \in C_r$, $d \in C_s$. Then $r, s \leq k$. If we put $\varphi_{k+1}(a) = \varphi_r(c)\varphi_s(d)$, then $\varphi = \bigcup\{\varphi_i :\ \ i \geq 0\}$ is a well defined mapping from H into G. Also, by induction on k we have: $\varphi(a^k) = (\varphi(a))^k$ and $\varphi(a^{(k)}) = (\varphi(a))^{(k)}$, for each $a \in H$ and $k \geq 0$.

If $a \in H$ is a 2-primitive element of $\boldsymbol{H}$ and $(c,d)|a$, then $\varphi(a) = \varphi(c)\varphi(d)$.

If $a \in H$ is such that $a = c^{(1)}$, $[c] = p$, $p \leq m-1$, then $\varphi(a) = \varphi(cc) = \varphi(c)\varphi(c)$.

If $c, d \in H$ are such that $c = d = a^{(p+m)}$, where $p \geq 0$, $a \in H$, then: $\varphi(cd) = \varphi(a^{(p+m+1)}) = \varphi((a^{(p+m)})^{(1)}) = (\varphi(a^{(p+m)}))^{(1)} = \varphi(c)\varphi(d)$.

If $c, d \in H$ are such that $c = a^{(p)}$, $d = a^{(p+m)}$, where $p \geq 0$, $a \in H$, then:
$\varphi(cd) = \varphi(a^{(p+m+1)}) = \varphi((a^{(p+m)})^{(1)}) = \varphi(a^{(p+m)})^{(1)} = \varphi(a)^{(p+m)}\varphi(a)^{(p+m)} =$
$(\varphi(a)^{(p)})^{(m)}(\varphi(a)^{(p)})^{(m)} = [\boldsymbol{G} \in \mathcal{V}_{(m)}] = (\varphi(a))^{(p)}(\varphi(a))^{(p+m)} =$
$= \varphi(a^{(p)})\varphi(a^{(p+m)}) = \varphi(c)\varphi(d)$.

Thus, in all possible cases we have $\varphi(cd) = \varphi(c)\varphi(d)$, i.e. φ is a homomorphism from $\boldsymbol{H}$ into $\boldsymbol{G}$. Therefore, $\boldsymbol{H}$ is a free groupoid in $\mathcal{V}_{(m)}$ with a free basis P. □

We will give an example of a $\mathcal{V}_{(m)}$-injective groupoid that is not free in $\mathcal{V}_{(m)}$. Let A be an infinite set and $H = A \times \mathbb{N}_0$ ($\mathbb{N}_0$ is the set of nonnegative integers). We will denote the elements of H by a_n instead of (a,n). Define a partial operation $\bullet$ on H by:

(i) $a_p \bullet a_p = a_{p+1}$, (ii) $a_p \bullet a_{p+m} = a_{p+m+1}$,
for any $p \geq 0$ and a fixed positive integer m.

Define a set $D \subseteq H \times H$ by:

$$D = \{(a_k, b_n) : a, b \in A\ \ \&\ \ k, n \in \mathbb{N}_0\ \ \&\ \ (a \neq b \vee (a = b\ \ \&\ \ k \neq p\ \ \&\ \ n \neq p\ \ \&\ \ n \neq p+m, p \geq 0))\}$$

Since $D \sim A \times \{0\}$, there is an injection $\varphi : D \to A \times \{0\}$ and we can put

(iii) $(\forall (a_k, b_n) \in D)\ \ a_k \bullet b_n = (\varphi(a_k, b_n))_0$.

By a direct verification we obtain that $(H, \bullet)$ is $\mathcal{V}_{(m)}$-injective groupoid. If φ is a bijection, then the set of primes in $\boldsymbol{H}$, i.e. $A \times \{0\} \setminus im\varphi$, is empty. Therefore, by the Bruck Theorem for $\mathcal{V}_{(m)}$, it follows that $(H, \bullet)$ is not free in $\mathcal{V}_{(m)}$. This and Prop.3.1. proves the following

PROPOSITION 3.2.. *The class of free groupoids in $\mathcal{V}_{(m)}$ is a proper subclass of the class of $\mathcal{V}_{(m)}$-injective groupoids.* □

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