ON A CLASS OF n-GROUPOIDS

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Abstract: Using the notion of vector valued semigroups, i.e $(m+k, m)$–semigroups ([1], [2]), a special class of n–groupoids, named "$m|k$–semigroups" is introduced and some examples of $m|k$–semigroups are given. It is shown that the general associative law (GAL) for $m|k$–semigroups holds, and some consequences of GAL are obtained. A description of the universal semigroup of an $m|k$–semigroup is given. The notion of $m|k$–group is also introduced and some properties are shown.

1. Introduction

Investigation of vector valued semigroups and groups, has led us to a class of $m|k$–semigroups.

First, we will introduce some notations which will be used further on:

1) The elements of Q^s, where Q^s denotes the s–th Cartesian power of Q, will be denoted by x_1^s.

2) The symbol x_i^j will denote the sequence $x_i x_{i+1} \ldots x_j$ when $i \leq j$, and the empty sequence when $i > j$.

3) If $x_1=x_2=\ldots=x_s=x$ $(x_i \in Q)$, then x_1^s is denoted by the symbol $\overset{s}{x}$.

If n, $m \in N$, where N is the set of positive integers, then an (n, m)–*operation* on a nonempty set Q is any mapping $f: Q^n \to Q^m$. In this case, we call the pair Q = (Q; f) an (n, m)–*groupoid*.

Let f be an (n, m)–operation on a set Q. We can associate to f a sequence of m n–ary operations $f_1, f_2, \ldots, f_m$ $(f_i : Q^n \to Q,\ 1 \leq i \leq m)$, by putting

$$f_i(a_1, a_2, \ldots, a_n) = b_i \Leftrightarrow f(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_m) \qquad (1)$$

for every $1 \leq i \leq m$.

Then, we call f_i the *i–th component operation of f* and we write:

$$f = (f_1, f_2, \ldots, f_m) \qquad (2)$$

Conversely, if $f_1, f_2, \ldots, f_m$ is a sequence of m n–ary operations on the set Q, then there is a unique (n, m)–operation f on Q, such that (2) holds.

If the equality (2) is true, we say that cp**Q** = (Q; $f_1, f_2, \ldots, f_m$) is the *component algebra of* **Q** and (Q; f_i) is the *i–th component n–groupoid of* **Q**.

Further on we will assume that Q≠∅ and that n, m are positive integers such that $n - m = k \geq 1$. An (n, m)–groupoid **Q** = (Q; f) is called an *(n, m)–semigroup* iff the following equality:

$$f(f(x_1^n)\, x_{n+1}^{n+k}) = f(x_1^i f(x_{i+1}^{i+n})\, x_{i+n+1}^{n+k}) \tag{3}$$

is an identity in (Q; f), for every $0 \leq i \leq k$.

Using the component algebra, (3) is equivalent to

$$f_j(f_1(x_1^n) \ldots f_m(x_1^n) x_{n+1}^{n+k}) = f_j(x_1^i f_1(x_{i+1}^{i+n}) \ldots f_m(x_{i+1}^{i+n}) x_{i+n+1}^{n+k}) \quad (3')$$

for every $0 \leq i \leq k$, $1 \leq j \leq m$.

The object of our interest is the case where component operations are mutually equal ($f_1 = f_2 = \ldots = f_m = f$), i.e a vector valued groupoid **Q** = (Q; f^*), where

$$f^* = (\underbrace{f, f, \ldots, f}_{m}).$$

Definition. Let $f: \mathrm{Q}^n \to \mathrm{Q}$, ($f: \mathrm{Q}^{m+k} \to \mathrm{Q}$). Then (Q; f) is an n–groupoid (m+k–groupoid) and we say that (Q; f) is an *m|k–groupoid*.

Definition. Let $f: \mathrm{Q}^{m+k} \to \mathrm{Q}$ be such that:

$$f((f(x_1^{m+k}))^m\, x_{m+k+1}^{m+2k}) = f(x_1^i (f(x_{i+1}^{i+m+k}))^m\, x_{i+m+k+1}^{m+2k}) \tag{4}$$

for any i, $0 \leq i \leq k$. Then, we say that (Q; f) is an *m|k–semigroup*.

We denote by $V_{m|k}$ the class of $m|k$–semigroups.

Remark. Obviously, if (Q; f) ∈ $V_{m|k}$ then (Q; f^*) is an (n, m)–semigroup, where $f^* = (\underbrace{f, f, \ldots, f}_{m})$. The convers is not true. Namely, let Q = $\boldsymbol{N}$ and $g: \boldsymbol{N}^5 \to \boldsymbol{N}^3$ be defined by $g(x_1^5) = (1, 1, x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5)$. Then, ($\boldsymbol{N}$; g) is a (5,3)–semigroup. If $f = g_3$, then it is easy to check that ($\boldsymbol{N}$; f) is not a 3|2–semigroup.

Examples of $m|k$–semigroups:

Example 1. Let Q ≠ ∅. Fix an element $a \in$ Q and put $f(x_1^{m+k}) = a$, for every $x_1^{m+k} \in \mathrm{Q}^{m+k}$. Then (Q; f) is an $m|k$–semigroup, called a *constant m|k – semigroup* on Q.

Example 2. Let m = 2, k = 2. Define a 2|2–operation f on $\boldsymbol{Q}$ by $f(x_1^4) = \frac{1}{2}(x_1 + x_2 + x_3 + x_4)$, for every $x_1^4 \in \boldsymbol{Q}^4$, where $\boldsymbol{Q}$ is the set of rational numbers. Then ($\boldsymbol{Q}$; f) is a 2|2–semigroup.

Example 3. Let $\boldsymbol{Q}$ be the set of rational numbers and define an $m|k$–operation f on $\boldsymbol{Q}$ by $f(x_1^{m+k}) = \frac{1}{m}(x_1 + x_2 + \ldots + x_{m+k})$, for every $x_1^{m+k} \in \boldsymbol{Q}^{m+k}$. Then ($\boldsymbol{Q}$; f) is an $m|k$–semigroup.

2. The general associative law (GAL)

We will show here that the "general associative law" is also true for $m|k$–semigroups. First we will introduce the notion of polynomial operation.

Let $(Q; f) \in V_{m|k}$. For $s \geq 1$, we define an operation $f^s: Q^{m+sk} \to Q$, inductively, in the following way:

$$1)\ f^1(x_1^{m+1k}) = f(x_1^{m+k})$$

$$2)\ f^{s+1}(x_1^{m+(s+1)k}) = f((f^s(x_1^{m+sk}))^m x_{m+sk+1}^{m+(s+1)k}) \qquad (5)$$

for every $x_1^{m+k} \in Q^{m+k}$ and $x_1^{m+(s+1)k} \in Q^{m+(s+1)k}$.

By induction on both r and s (as it is done in [2] for (n,m)–semigroups) we obtain the following statement:

Theorem. **(GAL)** If $(Q; f) \in V_{m|k}$, and $r, s \geq 1$, $x_1^{m+(r+s)k} \in Q^{m+(r+s)k}$, then

$$f^{r+s}(x_1^{m+(r+s)k}) = f^r(x_1^i(f^s(x_{i+1}^{i+m+sk}))^m x_{i+m+sk+1}^{m+(r+s)k}) \qquad (6)$$

for every $0 \leq i \leq rk$.

3. Some consequences of GAL

As a direct consequence of GAL we have the following proposition:

***Proposition* 3.1** If $(Q; f) \in V_{m|k}$, then $(Q; f^s) \in V_{m|sk}$, for every $s \geq 1$.

Proof. Let $(Q; f)$ be an $m|k$–semigroup. Then,

$$f^s((f^s(x_1^{m+sk}))^m x_{m+sk+1}^{m+2sk}) = f^{2s}(x_1^{m+2sk}).$$

By using GAL, we obtain $f^{2s}(x_1^{m+2sk}) = f^s(x_1^i(f^s(x_{i+1}^{i+m+sk}))^m x_{i+m+sk+1}^{m+2sk})$, where $0 \leq i \leq sk$. This, implies that $(Q; f^s)$ is an $m|sk$–semigroup, for every $s \geq 1$.

The notion of a commutative $m|k$–groupoid can also be introduced in an obvious way.

Definition. An $m|k$–groupoid $(Q; f)$ is *commutative* iff for every permutation $\sigma \in S_{m+k}$ the following identity holds:

$$f(x_1^{m+k}) = f(\sigma(x_1^{m+k})) \qquad (7)$$

where $\sigma(x_1^{m+k}) = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(m+k)}$.

Using GAL and the usual induction, we obtain:

***Proposition* 3.2** If $(Q; f)$ is a commutative $m|k$–semigroup, then $(Q; f^s)$ is a commutative $m|sk$–semigroup, for every $s \geq 1$.

Definition. An $m|k$–groupoid $(Q; f)$ is *cancellative* iff every implication

$$f(a_1^i \overset{m}{x} a_{i+1}^k) = f(a_1^i \overset{m}{y} a_{i+1}^k) \Rightarrow x = y \tag{8}$$

holds in (Q; f), for every $0 \le i \le k$.

Now we give the following characterization for the cancellative $m|k$–semigroups. The proof is by analogy to the proof of Theorem 5.7. in [2].

***Proposition* 3.3** If (Q; f)$\in V_{m|k}$, then the following conditions are equivalent:

(i) (Q; f) is a cancellative $m|k$–semigroup;
(ii) (Q; f^s) is a cancellative $m|sk$–semigroup, for every $s \ge 1$;
(iii) (Q; f^s) is a cancellative $m|sk$–semigroup, for some $s \ge 1$;
(iv) (Q; f) satisfies the following implication:

$$[f(a_1^k \overset{m}{x}) = f(a_1^k \overset{m}{y}) \vee f(\overset{m}{x} b_1^k) = f(\overset{m}{y} b_1^k)] \Rightarrow x = y \tag{9}$$

(v) There are i, $s \ge 1$, $i \in N_{sk-1}$, $sk \ge 2$ such that the following implication in (Q; f^s) is true:

$$f^s(a_1^i \overset{m}{x} a_{i+1}^{sk}) = f^s(a_1^i \overset{m}{y} a_{i+1}^{sk}) \Rightarrow x = y \tag{10}$$

Remark. An (n,m)–semigroup **Q** is said to be cancellative (see [2]) if

$$f(a_1^i x_1^m a_{i+1}^k) = f(a_1^i y_1^m a_{i+1}^k) \;\Rightarrow\; x_i = y_i, i = 1, \ldots, m$$

for any i, $0 \le i \le k$.

This definition can be used as a definition for cancellative $m|k$–semigroup, but the one given in this paper is not convenient in defining, for example, $m|k$–groups.

Definition. An $m|k$–semigroup (Q; f) is called an $m|k$–*group* , iff

$$(\forall a_1^k \, \mathrm{Q}^k,\ b \in \mathrm{Q})\ (\exists x, y \in \mathrm{Q})\ f(a_1^k \overset{m}{x}) = b = f(\overset{m}{y} a_1^k). \tag{11}$$

Example. Let $f: \boldsymbol{Q}^{m+k} \to \boldsymbol{Q}$ be defined by $f(x_1^{m+k}) = \frac{1}{m}(x_1 + x_2 + \ldots + x_{m+k})$, for every $x_1^{m+k} \in \boldsymbol{Q}^{m+k}$, where $\boldsymbol{Q}$ is the set of rational numbers. Then ($\boldsymbol{Q}$; f) is an $m|k$–group.

The proposition bellow gives a characterization of $m|k$–groups ([2]).

***Proposition* 3.4** If (Q; f)$\in V_{m|k}$, then the following conditions are equivalent:

(i) (Q; f) is an $m|k$–group;
(ii) (Q; f^s) is an $m|sk$–group, for every $s \ge 1$;
(iii) (Q; f^s) is an $m|sk$–group, for some $s \ge 1$;
(iv) There is an $s \ge 1$, satisfying $sk \ge 2$ and $i \in N_{sk-1}$, such that the equation:

$$f^s(a_1^i \overset{m}{x} a_{i+m+1}^{m+sk}) = b \tag{12}$$

is solvable for given $a_1^i \in \mathrm{Q}^i$, $a_{i+m+1}^{m+sk} \in \mathrm{Q}^{sk-i}$, $b \in$ Q;

(v) There is an $s \ge 1$, such that $sk \ge 2$ and for each $i \in N_{sk-1}$, the equation (12) is solvable;

(vi) For each $s \ge 1$, satisfying $sk \ge 2$, there is an $i \in N_{sk-1}$, such that the equation (12) is solvable;

(vii) For each $s \geq 1$, satisfying $sk \geq 2$, and each $i \in N_{sk-1}$, the equation (12) is solvable.

4. Universal semigroup of an m| k – semigroup

In this section we will consider the question for embedding a given $m|k$–semigroup (Q; f) into a semigroup, and, we will give a description of the universal semigroup of an $m|k$–semigroup (Q; f).

Let (Q; f) be an $m|k$–semigroup and let $\Lambda = \{(a_1^{m+k}, \overset{m}{b}) \mid f(a_1^{m+k}) = b\}$. We denote by Q^+ the free semigroup with a basis Q. If $\approx$ is the least congruence on the semigroup Q^+ containing Λ, then quotient groupoid $Q^+/\approx$ with the operation $\cdot$ defined by:

$$a_1^s \cdot a_{s+1}^t = \begin{cases} a_1^t & , \ t < m+k \\ (f(a_1^{m+k}))^m \cdot a_{m+k+1}^t & , \ t \geq m+k \end{cases} \tag{13}$$

for every $a_\upsilon \in Q$, is a semigroup.

We denote by Q^ the semigroup $Q^+/\approx$, (Q^ = $Q^+/\approx$) and we write **Q^** = (Q^, $\cdot$). Then the following propositions hold:

1. $f(a_1^{m+k}) = b$ in **Q** $\Rightarrow$ $a_1 \cdot a_2 \cdot \ldots \cdot a_{m+k} = \overset{m}{b}$ in **Q^**.
2. $a, b \in Q$, $a \approx b \Rightarrow a = b$.

Let $\Delta = \{(\overset{m}{b}, b) \mid (\exists\, a_1, \ldots, a_{m+k} \in Q)\ a_1 \cdot a_2 \cdot \ldots \cdot a_{m+k} = \overset{m}{b}\}$, and let $\sim$ be the least congruence on Q^ containing Δ. Then, Q^/$\sim$ with the operation $*$ defined by:

$$a_1^s * a_{s+1}^t = \begin{cases} a_1^t & , \ t < m+k \\ f(a_1^{m+k}) * a_{m+k+1}^t & , \ t \geq m+k \end{cases} \tag{14}$$

for every $a_\upsilon \in Q$, is a semigroup.

Let Q^Δ = Q^/$\sim$, and $\mathbf{Q}^\Delta = (Q^\Delta; *)$. Then we have the following:

1' $f(a_1^{m+k}) = b$ in **Q** $\Rightarrow$ $a_1 * a_2 * \ldots * a_{m+k} = b$ in $\mathbf{Q}^\Delta$.

2' $a, b \in Q$, $a \sim b$ $\Rightarrow$ $a = b$.

In other words, by 2' we can assume that Q is a subset of Q^Δ.

For the class of $m|k$–semigroups, we have the following result.

Theorem: If **P** =(P; $\bullet$) is a semigroup, such that $Q \subseteq P$ and

$$f(a_1^{m+k}) = b \text{ in } \mathbf{Q} \ \Rightarrow\ [a_1 \bullet a_2 \bullet \ldots \bullet a_{m+k} = \overset{m}{b} \ \wedge\ a_1 \bullet a_2 \bullet \ldots \bullet a_{m+k} = b] \text{ in } \mathbf{P}.$$

Then the inclusion $a \mapsto a$ of Q into P can be uniquely extended to a homomorphism from $\mathbf{Q}^\Delta$ into **P**.

Proof. Let $\varphi\colon Q^{\Delta} \to P$ be defined by:

$$\varphi(a) = a, \qquad \varphi(a_1^t) = a_1 \bullet a_2 \bullet \ldots \bullet a_t$$

for every $a \in Q$ and $a_1^t \in Q^{\Delta}$.

We will show that φ is a well–defined mapping. Let $a_1^t = b_1^s$.

1) If $a_1^t = a_1^i \overset{m}{a_{i+1}} a_{i+m+1}^t$, $b_1^s = a_1^i a_{i+1} a_{i+m+1}^t$, and, there is $c_1^{m+k} \in Q^{m+k}$, such that $f(c_1^{m+k}) = a_{i+1}$, then:

$$\varphi(a_1^t) = a_1 \bullet a_2 \bullet \ldots \bullet a_i \bullet \overset{m}{a_{i+1}} \bullet \ldots \bullet a_t = a_1 \bullet a_2 \bullet \ldots \bullet a_i \bullet c_1 \bullet \ldots \bullet c_{m+k} \bullet \ldots \bullet a_t$$

$$= a_1 \bullet a_2 \bullet \ldots \bullet a_i \bullet a_{i+1} \bullet \ldots \bullet a_t = \varphi(b_1^s).$$

2) Let $a_1^t = u_0 \approx u_1 \approx \ldots \approx u_r = b_1^s$. Then, by 1),

$$\varphi(a_1^t) = \varphi(u_0) = \varphi(u_1) = \ldots = \varphi(u_r) = \varphi(b_1^s).$$

Now, we will show that φ is a homomorphism. Let $a_1^t, a_{t+1}^s \in Q^{\Delta}$.

1) $s < m+k$

$$\varphi(a_1^t * a_{t+1}^s) = \varphi(a_1^s) = a_1 \bullet a_2 \bullet \ldots \bullet a_t \bullet a_{t+1} \bullet \ldots \bullet a_s = \varphi(a_1^t) \bullet \varphi(a_{t+1}^s).$$

2) $s \geq m+k$. Then, $\varphi(a_1^t * a_{t+1}^s) = \varphi(c * a_{m+k+1}^s)$, where $c = f(a_1^{m+k})$. We can assume that $s-m-k+1 < m+k$ and then, we have:

$$\varphi(c * a_{m+k+1}^s) = \varphi(c a_{m+k+1}^s) = c \bullet a_{m+k+1} \bullet \ldots \bullet a_s = a_1 \bullet a_2 \bullet \ldots \bullet a_t \bullet a_{t+1} \bullet \ldots \bullet a_{m+k} \bullet a_{m+k+1} \bullet \ldots \bullet a_s$$

$$= \varphi(a_1^t) \bullet \varphi(a_{t+1}^s).$$

In other words, $\mathbf{Q}^{\Delta}$ is the universal semigroup of $\mathbf{Q} = (Q; f)$.

As a consequence of the last theorem, we obtain the Post's Theorem for $m|\,k$–semigroups, i.e. *every $m|\,k$–semigroup can be embedded in a semigroup.*

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