ON A VARIETY OF GROUPOIDS OF RANK 1

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Abstract: A canonical description of free objects in the variety \( \mathcal{V} \) of groupoids defined by the identity \( xx^2 = x^2x^2 \) is given. Injective groupoids in \( \mathcal{V} \) and subgroupoids of free groupoids in \( \mathcal{V} \) are considered. It is shown that the class of free groupoids in \( \mathcal{V} \) is a proper subclass of the class of injective groupoids in \( \mathcal{V} \) and that both classes are hereditary.

0. Introduction

Throughout the paper we denote by \( \mathcal{F} = (F, \cdot, ) \) a given absolutely free groupoid with a basis \( B \) (i.e. groupoid free in the class of all groupoids). It is well-known (1, [1, L.1.5]) that the following two conditions characterize \( \mathcal{F} \): a) \( \mathcal{F} \) is injective ; b) the set \( B \) of prime elements in \( \mathcal{F} \) is nonempty and generates \( \mathcal{F} \).

The subject of this paper is the variety of groupoids of rank 1, \(^{2}\) defined by the identity \(^{3}\)

\[ xx^2 = x^2x^2, \quad (0.1) \]

which we denote by \( \mathcal{V} \). The paper is divided into three sections.

In Section 1 we give a description of \( \mathcal{V} \)-free groupoids and show that they may be different.

In Section 2, the notion of \( \mathcal{V} \)-injective groupoid (i.e. groupoid injective in \( \mathcal{V} \)) is introduced. It is shown that the class of \( \mathcal{V} \)-injective groupoids is hereditary, every \( \mathcal{V} \)-injective groupoid is infinite and the class of \( \mathcal{V} \)-free groupoids is a proper subclass of the class of injective groupoids.

In Section 3 are considered subgroupoids of \( \mathcal{V} \)-free (i.e. free in \( \mathcal{V} \)) groupoids. We prove that: Bruck Theorem for \( \mathcal{V} \) holds, the class of \( \mathcal{V} \)-free groupoids is hereditary and that every \( \mathcal{V} \)-free groupoid contains a subgroupoid with an infinite basis.

1. Free objects in \( \mathcal{V} \)

For a construction of a free object in a variety \( \mathcal{V} \) of groupoids with an axiom \( f = g \), the following "procedure" is often convenient. We consider one of the two parts of the axiom of \( \mathcal{V} \) as "more suitable", and, as a candidate for the carrier of the desired free object, we choose the set \( R \) of all elements \( t \in \mathcal{F} \) which contain no parts with a form of the "unsuitable side of the axiom".

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1) We do not define here the notions as injective groupoid, prime element, length \( |v| \) and set \( P(v) \) of parts of \( v \in \mathcal{F} \) \( (\text{see, for example} \ [2]) \).

2) A variety defined by identities with contain only one variable is called a variety of rank 1.

3) Here we use the usual abbreviations: \( xx^2 = x^4, x^2x^2 = (xx)(xx) \).
Here we consider the variety $\mathcal{V}$ with an axiom $x x^2 = x^2 x^2$. Choosing the left side as "unsuitable", we obtain:

\[ R = \{ t \in F : (\forall \alpha \in F) \alpha \alpha^2 \not\in P(t) \} \quad \text{(1.1)} \]

By (1.1) immediately we obtain:

a) $(\forall t, u \in F) \{ t, u \in R \iff t, u \in R \& u \neq t \}$.

b) $t, u \in R \Rightarrow \{ tu \in R \iff u = t \}$.

c) $t \in R \Rightarrow (\forall k \in \mathbb{N}) t^k \in R \quad \text{(1.0)}\) $(t^n$ is defined in $F$ by: $t^1 = t$, $t^{n+1} = t^n \ast t$).

Define an operation $\ast$ on $R$ by:

\[ t, u \in R \Rightarrow t \ast u = \begin{cases} tu, & \text{if } tu \in R \\ (t^2)^2, & \text{if } u - t^2 \end{cases} \quad \text{(1.2)} \]

By a direct verification we obtain that $R = (R, \ast)$ is a groupoid, the equality (0.1) holds in $R$ (i.e. $t^i(t^i t) = (t^i t)^i$ is an identity in $R$) which means that $R \in \mathcal{V}$ and for any mapping $\lambda : B \to G$\(^7\), there is a homomorphism $\phi : F \to G$ which extends $\lambda$. Therefore:

**Theorem 1.** $R = (R, \ast)$ is a free groupoid in $\mathcal{V}$ with the basis $B$, and $B$ coincides with the set of primes in $R$.

Below we will state some properties of the groupoid $R$, but first we will consider the groupoid power $x^{(k)}$, $k \geq 0$, defined by:

\[ x^{(0)} = x, \quad x^{(k+1)} = x^{(k)} x^{(k)} = (x^{(k)})^2 \quad \text{(1.3)} \]

By induction on $m$ and $n$ one can show that, in any groupoid $G = (G, \cdot)$, the following statement is true:

\[ (\forall x \in G, m, n \geq 0) \quad (x^{(m)})^{(n)} = x^{(m+n)} \quad \text{(1.4)} \]

If $G \in \mathcal{V}$, then by (1.3) and (0.1) one obtains:

\[ (\forall x \in G, p \geq 0) \quad x^{(p)} = x^{(p+1)} = x^{(p+2)} \quad \text{(1.5)} \]

We say that an element $a \in G$ is a power in $G$ if there are $b \in G$ and $k \geq 1$, such that $a = b^k$. If $b \in G$ is not a power in $G$, i.e.

\[ (\forall c \in G) \quad (c = b^{(p)} \Rightarrow p = 0) \]

then we say that $b$ is a base in $G$.

As a special case of c), one obtains:

\[ c') t \in R, k \geq 1 \Rightarrow t^{(k)} \in R. \]

Note that, if $t \in R$, $t^n$ is the $n$-th power of $t$ in $F$; in this sense, $t_n$ is the $n$-th power of $t$ in $R$, defined by: $t_1 = t$, $t_{n+1} = t_n \ast t$.

From (1.2) and c'), by induction on $k$, we obtain:

\[ P(t) \text{ is the set of parts (i.e. subterms) of } t. \]

\[ N \text{ is the set of positive integers.} \]

\[ \text{The carrier of a given groupoid } S \text{ is denoted by the same (light) letter } S. \]
\[ (\forall t \in R, \ k \geq 0) \ \ i_k^t = t^k \] and, for \( k \geq 1 \): \[ t_k = t^k. \]

As a consequence of d), we obtain the following two statements.

**Proposition 1.1.** \((\forall u \in R)(\exists (i, p) \in R \times N_0) u = i^p\), where \( i \) is a base in \( R \).

**Proposition 1.2.**

a) If \( x \in R \setminus B \) (i.e. \( x \) is not prime in \( R \)) and if \( x \) is not a power, then there exists a unique pair \( (u, v) \in R^2 \), such that \( x = u \cdot v \). (In this case \( x = uv \).)

We say that \( (u, v) \) is the pair of divisors of \( x \) and write \( (u, v) | x \).

b) If \( x \in R \) is a power, \( x = i^{(p+1)} \), \( p \geq 0 \), then \( x = i^p \cdot i^{(p)} \), and \( (i^p, i^{(p)}) \) is the pair of divisors of \( x \).

The class of groupoids free in \( V \) will be denoted by \( V_{\text{free}} \). We will show the following

**Proposition 1.3.** If \( H \in V_{\text{free}} \) with the basis \( B \), then there exists a mapping \( x \mapsto |x| \) from \( H \) into \( N \) such that

\[
|b| = 1, \quad |cd| \geq |c| + |d|, \quad (1.6)
\]

for any \( b \in B, \ c, d \in H \).

**Proof.** Let \( R = (R, *) \) be the \( V \)-canonical groupoid with the basis \( B \) (constructed above, Th.1) and let \( x \mapsto |x| \) be the restriction of the mapping \( | \cdot : F \to N \).

Let \( i, u \in R \). Since \( i, u \in F \), it follows that \( |tu| = |t| + |u| \) and \( |it^2| = 3|t| \). From (1.2) we obtain:

\[
u t \in R \Rightarrow \ t * u = |t| + |u|, \quad (1.2) \]
\[
u t \in R \Rightarrow u = t^2 \Rightarrow t * t^2 = (t^2)^2 = 4 |t| > 3 |t|.
\]

This shows that \( |t * u| \geq |t| + |u| \).

Now, let \( H \in V_{\text{free}} \) with the basis \( B \). Since \( H \) is isomorphic with \( R \), it follows that (1.6) holds.

**Remark 1.4.** When one decides which side of the identity \( xx^2 = x^2 x^2 \) to consider as "suitable" one, it is natural to choose the "shorter" side, i.e. \( xx^2 \), and expect a shorter construction of \( V \)-free groupoid. However, it turns out the opposite, the construction is longer and more complicated.

Namely, let the first candidate for the carrier of \( V \)-free groupoid be the set \( F_i \), defined by:

\[
F_i = \{ t \in F : (\forall \alpha \in F) \ (\alpha^2)^2 \in P(t) \}.
\]

\( N_0 \) is the set of nonnegative integers.
If we define an operation \( *_1 \) on \( F_1 \) by:
\[
t, u \in F_1 \Rightarrow t *_1 u = \begin{cases} 
u, & \text{if } u \in F_1 \\ \alpha \alpha^2, & \text{if } t = u = \alpha^2 \end{cases}
\]
then we obtain that \( F_1 = (F_1, *_1) \) is a groupoid. However, the equality (0.1), which has the form here
\[
t *_1 (t *_1 t) = (t *_1 t) *_1 (t *_1 t)
\]  
(1.7)
is not satisfied in \( F_1 \), i.e. \( F_1 \not\in \mathbb{V} \). Namely, if \( t = \alpha^2 \), the left side of (1.7) is \( \alpha^2 (\alpha \alpha^2) \) and the right one is \( (\alpha \alpha^2)^2 \). This result implies that
\[
\alpha^2 (\alpha \alpha^2) = (\alpha \alpha^2)^2
\]  
(1.7)
is an identity in \( \mathbb{V} \). This suggests a definition of a new "candidate" \( F_2 = (F_2, *_2) \):
\[
F_2 = \{ t \in F_1 : (\forall \alpha \in F_1) (\alpha (\alpha \alpha^2)^2 \not\in P(t)) \}
\]
\[
t, u \in F_2 \Rightarrow t *_2 u = \begin{cases} 
\alpha \alpha^2, & \text{if } t *_1 u \in F_2 \\
\alpha^2 (\alpha \alpha^2), & \text{if } t = u = \alpha^2
\end{cases}
\]
Checking (1.7) (when \( *_1 \) is substituted by \( *_2 \)), we obtain \( F_2 \not\in \mathbb{V} \) and one more identity in \( \mathbb{V} \):
\[
(\alpha \alpha^2)^2 (\alpha^2 (\alpha \alpha^2)^2) = ((\alpha^2 (\alpha \alpha^2)^2))^2.
\]  
(1.7′)
Continuing this procedure, we can see a regularity in the consequences of the identity (1.7), which suggests to introduce a special kind of groupoid power \( x \mapsto x^{<n>} \), defined by:
\[
x^{<0>} = x, \quad x^{<1>} = x^2, \quad x^{<k+2>} = x^{<k>} x^{<k+1>}. \]  
(1.8)

Using this, we define the following infinite set of groupoids:
\[
\{ F_n = (F_n, *_{n+1}) : n \geq 1 \},
\]
\[
F_1 = \{ t \in F : (\forall \alpha \in F) (\alpha (\alpha \alpha^2)^2 \not\in P(t)) \}
\]
\[
t, u \in F_1 \Rightarrow t *_1 u = \begin{cases} 
u, & \text{if } u \in F_1 \\ \alpha \alpha^2, & \text{if } t = u = \alpha^2
\end{cases}
\]
\[
F_{n+1} = \{ t \in F_n : (\forall \alpha \in F_n) (\alpha (\alpha \alpha^2)^2 \not\in P(t)) \}
\]
\[
t, u \in F_{n+1} \Rightarrow t *_{n+1} u = \begin{cases} 
\alpha \alpha^2, & \text{if } t *_1 u \in F_{n+1} \\
\alpha^2 (\alpha \alpha^2), & \text{if } t = u = \alpha^2
\end{cases}
\]
One can show that \( F_{n+1} \) is a groupoid such that \( F_{n+1} \not\in \mathbb{V} \).
Using the fact that \( F \supseteq F_1 \supseteq \ldots \supseteq F_n \supseteq \ldots \) and that \( F_{n+1} \) is "better" than \( F_n \), we obtain the following definition of the carrier \( R' \) of a free groupoid in \( \mathbb{V} \):
\[
R' = \{ t \in F : (\forall \alpha \in F_n, n \geq 1) (\alpha (\alpha \alpha^2)^2 \not\in P(t)) \} \setminus \{ F_n : n \geq 1 \}
\]
(Note that it is not necessary to define the whole sequence, because the desired "good candidate" can be noticed after several steps.)

We define an operation \( \ast \) on \( R' \) by
\[ t, u \in R' \Rightarrow t + u = \begin{cases} tu, & \text{if } tu \in R' \\ \alpha^{<n+1>}, & \text{if } t = u = \alpha^{<n>}, \; n \geq 1 \end{cases} \]

and obtain:

**Proposition 1.5.** \( R' \) is a \( \forall \)-free groupoid with the basis \( B \).

Therefore, there exist at least two distinct \( \forall \)-free groupoids, \( R \) and \( R' \). Since \( R \) and \( R' \) have the same basis \( B \), they are isomorphic.

2. **Injective groupoids in \( \forall \)**

We obtain the class of injective groupoids in \( \forall \) from the class of \( \forall \)-free groupoids in the same way as in the variety of all groupoids (and, namely, by omitting the condition that the set of primes is a generating set).

Using Pr.1.1 - 1.2, we come to the following definition of injective groupoids in \( \forall \).

We say that a groupoid \( H = (H, \cdot) \in \forall \) is injective in \( \forall \) (i.e. \( \forall \)-injective) iff the following conditions are satisfied:

\( i_i \) (\( \forall a \in H \) (\( \exists (b, p) \in H \times N_0 \)) \( a = b^{(p)} \) and \( b \) is a base in \( H \).

(We say that \( b \) is the base and \( p \) the exponent of \( a \).)

\( i_i \) \( b^{(p+2)} = cd \) iff \( c = d = b^{(p)} \) or \( c = b^{(p)} \) and \( d = b^{(p+1)} \).

\( i_i \) If \( b \) is a base in \( H \), then: \( b^{(1)} - cd \Leftrightarrow c = d = b \).

\( i_i \) If \( a \in H \) is a base which is not prime in \( H \), then

\( \exists \text{?} (c, d) \in H^2 \).

The pair of divisors \( (c, d) \) of an element \( a \in H \) which is not prime in \( H \) (we write: \( (c, d) \mid a \) ) is defined as follows.

1) If \( b \) is a base in \( H \) and \( p \geq 0 \), then \( (b^{(p)}, b^{(p+1)}) \mid b^{(p+1)} \).

2) If \( a = cd \) is a base in \( H \), then \( (c, d) \mid a \).

We denote by \( \forall_{\text{inj}} \) the class of injective groupoids. By Pr.1.1-1.2 we obtain:

**Proposition 2.1.** \( \forall_{\text{free}} \subset \forall_{\text{inj}} \).

**Proposition 2.2.** Let \( H \in \forall_{\text{inj}} \), \( Q \leq H \), \( a = b^{(p)} \in Q \) and \( b \notin Q \), where \( b \) is the base of \( a \) in \( H \). If \( r = \min \{ k; b^{(k)} \in Q \} \), then \( b^{(r)} \) is a prime in \( Q \).

As a corollary of Pr.2.2, we obtain the following

**Proposition 2.3.** The class \( \forall_{\text{inj}} \) is hereditary.

If \( c^{(p)} = d^{(q)} \) and \( c, d \) are bases in \( H \in \forall_{\text{inj}} \), then \( c = d \) and \( p = q \). Therefore, if \( a \) is a base in \( H \), then the powers \( a^{(n)} \), \( n \geq 1 \), are mutually distinct and thus the set \( \{a^{(1)}, a^{(2)}, \ldots \} \) is infinite. Therefore:
Proposition 2.4. Every $H \in \mathcal{V}_{\text{im}}$ is infinite.

Below we will give a construction of $\mathcal{V}$-injective groupoids (and show that there are $\mathcal{V}$-injective groupoids which are not $\mathcal{V}$-free).

Let $A$ be an infinite set and $H = A \times \mathbb{N}_0$. Define a partial operation $\bullet$ on $H$ by:

$$(a, p) \bullet (a, p) = (a, p + 1),$$

$$(a, p) \bullet (a, p + 1) = (a, p + 2)$$

and put

$$D = \{(a, p), (b, q) : a \neq b \text{ or } (a = b \land q \in \{p, p + 1\})\}.$$

Since $A$ and $D$ have the same cardinality, there is an injection $\varphi : D \rightarrow A$ and we can put

$$(\forall (a, p), (b, q) \in D) \quad (a, p) \bullet (b, q) = (\varphi((a, p), (b, q)), 0).$$

Then we obtain that $(H, \bullet)$ is a groupoid injective in $\mathcal{V}$.

If $\varphi$ is a bijection, then the set $A \times \{0\} \setminus \text{im} \varphi$ of primes in $H$ is empty, and thus $(H, \bullet)$ is not free in $\mathcal{V}$.

This and Cor.2.1 proves the following

Theorem 2. The class $\mathcal{V}_{\text{free}}$ is a proper subclass of $\mathcal{V}_{\text{im}}$.

3. Subgroupoids of $\mathcal{V}$-free groupoids

In this section we will show that the class $\mathcal{V}_{\text{free}}$ is hereditary, but first we will give a characterization of $\mathcal{V}$-free groupoids (analogous to a), b) in Introduction, for absolutely free groupoids).

Theorem 3 (Bruck Theorem for $\mathcal{V}$). A groupoid $H \in \mathcal{V}$ is $\mathcal{V}$-free iff:

(i) $H$ is $\mathcal{V}$-injective.

(ii) The set $B$ of primes in $H$ is nonempty and generates $H$.

Proof. If $H$ is $\mathcal{V}$-free with the basis $B$, then $H \in \mathcal{V}_{\text{free}}$ (by Th.2), $B$ is the set of primes in $H$ and generates $H$.

For the converse, define an infinite sequence of subsets $B_1, B_2, \ldots$ of $H$:

$$B_1 = B, \quad B_2 = \{c^d : c, d \in B_1\},$$

$$B_{k+1} = \{a \in H : (c, d) \rightarrow (c, d) \subseteq B_1 \cup \ldots \cup B_k \land (c, d) \not\in \emptyset\}.$$

Then the following statements are true:

1) $(\forall k \geq 1) \ B_k \neq \emptyset$; 2) $p \neq q \rightarrow B_p \cap B_q = \emptyset$; 3) $H = \bigcup \{B_k : k \geq 1\}$.

Let $G \in \mathcal{V}$ and $\lambda : B \rightarrow G$ be a mapping. Defining a sequence of mappings $\varphi_k : B_1 \rightarrow G$ inductively ($\varphi_1 = \lambda; \ldots$) and continuing in a similar way as in [2, Th.1], one can show that the mapping $\varphi = \bigcup \{\varphi_k : k \geq 1\}$ is a homomorphism from $H$ into $G$ which extends $\lambda$.

Below we assume that $H \in \mathcal{V}_{\text{free}}$ and $Q$ is a subgroupoid of $H$.

Proposition 3.1. The set $P$ of primes in $Q$ is nonempty and generates $Q$.
Proof. By Pr.1.3, there is a mapping in $H$ with the property (1.6). Let $c \in Q$ be such that

$$|c| = \min \{|a| : a \in Q\}$$

(3.1)

By (1.6) and (3.1), $c$ is prime in $Q$. Thus the set $P$ of primes in $Q$ is nonempty.

Denote by $T$ the groupoid generated by $P$. Clearly, $T \subseteq Q$. We will show that $Q \subseteq T$, using induction on length.

First of all, $P \subseteq T$. If $b \in Q$ and $|b| = 1$, then $b$ is prime in $H$, and thus $b$ is prime in $Q$, i.e. $b \in P$. Therefore $b \in T$. Suppose $c \in Q$ & $|c| \leq k \Rightarrow c \in T$.

Let $c \in Q$ and $|c| = k + 1$. If $c \in P$, then $c \in T$. Therefore let $c \in Q \setminus P$. Then $c = de$ ($d, e \in Q$), and:

$$|c| = |d| \leq |d| + |e| \Rightarrow |d|, |e| \leq k \Rightarrow d, e \in T \Rightarrow c = de \in T.$$ 

Thus $Q \subseteq T$ and therefore $P$ generates $Q$.

Proposition 3.2. $Q \in V_{\text{free}}$.

Proof. By Pr.2.1, $H \in V_{\text{free}}$ implies $H \in V_{\text{inf}}$, and by Pr.2.3, $Q \in V_{\text{inf}}$. Now, applying Bruck Theorem for $V$ we obtain that $Q \in V_{\text{free}}$.

As a corollary of Pr.3.2 we obtain

Proposition 3.3. The class $V_{\text{free}}$ is hereditary.

Finally, we will prove the following

Proposition 3.4. If $H \in V_{\text{free}}$, then there is a subgroupoid $Q$ such that $Q$ has an infinite basis.

Proof. Let $B$ be the basis of $H$ and $a \in B$. Put

$$c_1 = a \cdot a, c_2 = a_1 a, \ldots, c_{k+1} = a_k a, \ldots$$

and let $Q$ be the subgroupoid of $H$ generated by the set $C = \{c_k : k \geq 0\}$. Then: 1) $C$ is infinite; 2) all the elements in $C$ are prime in $Q$; 3) $C$ is a basis of $Q$.

Namely, the elements of $C$ are mutually distinct ($c_p = c_q \Rightarrow p = q$) and thus $C$ is infinite. Secondly, every element $c_k \in C$ is a base in $Q$: $c_1 = aa = a^2$, but $a \notin Q$ and thus $c_1$ is a base in $Q$: $c_1 = a^2 a = a^3$, $a^2 \in Q$, but $a \notin Q$, and thus $c_2$ is a base in $Q$: inductively, $c_{k+1} = a^k a$ is a base in $Q$. Thirdly, $C$ is the set of primes in $Q$, $C \neq \emptyset$ and generates $Q$. Therefore, $C$ is the basis of $Q$.

References
