FREE POWER-SEMICOMMUTATIVE TERNARY GROUPOIDS

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Abstract. A description of free objects in the class $\mathcal{P}_{sc}$ of power-semi-commutative ternary groupoids and a proper characterization of the class of free objects in $\mathcal{P}_{sc}$ by means of $\mathcal{P}_{sc}$-injective ternary groupoids are obtained.

1. Introduction

Ternary algebraic operations were introduced in the 19th century by A. Cayley. In 1928, W. Dörnte, under inspiration of Emmy Noether, introduced the notion of an n-ary group. Later on, in 1932, D. H. Lehmer introduced the notion of ternary algebraic system by investigating certain algebraic systems called triplexes which turn out to be commutative ternary groups ([19]). Ternary structures have many applications in different branches, such as Lie algebra theory ([22]), matrix theory ([18]), differential geometry, theoretical and mathematical physics ([17, 20]), etc. Many generalizations of binary structures to higher n-ary contexts are considered by many authors. Free n-ary algebraic structures (specially free ternary algebraic structures) are investigated, for example, in [2, 9, 10, 14, 24].

Groupoids (i.e. algebras with one binary operation) with a power-property, such as power-associativity and power-commutativity, are structures that have a weak form of associativity ([12]) or commutativity ([6]). It seams that power-associativity have been first studied by Albert in 1948 ([1]), concerning rings. A groupoid is said to be power-associative (power-commutative) if the subgroupoid generated by any of its elements is a subsemigroup (commutative subgroupoid). The concept of power-property can be extended to n-ary groupoids ([11]) and specially to ternary groupoids.

By a ternary groupoid we mean a nonempty set G with one ternary operation $[\,] : G \times G \times G \to G$. For the notation and basic notions the reader is referred to [16] and [23].

A ternary groupoid $\boldsymbol{G} = (G, [\,])$ is said to be *semicommutative* ([3, 4, 15, 21]) if

$$(\forall a, b, c \in G) \ \ [abc] = [cba].$$

The class $\mathcal{S}_c$ of all semicommutative ternary groupoids is a variety, defined by the identity $[x_1 x_2 x_3] \approx [x_3 x_2 x_1]$.

Some properties of semicommutative ternary groups are considered in [4]. Namely, a ternary group is medial if and only if it is semicommutative, and a ternary group $\boldsymbol{G}$ is semicommutative if and only if there exists $a \in G$ such that $[xay] = [yax]$, for all $x, y \in G$. Semicommutative n-monoids are considered in [21].

Power-semicommutative ternary groupoids are ternary groupoids such that every ternary subgroupoid generated by any of its elements is a semicommutative ternary subgroupoid.

For example, let $G = \{0, 1, 2\}$ and let $[\,] : G^3 \to G$ be a ternary operation defined by:

$[000] = [010] = [101] = [011] = [110] = 1, \quad [111] = [001] = [100] = 0,$

$[xyz] = 2$, if $y = 2$ or $z = 2$ and $[2yz] = 1$, for every $y, z \in G$.

Clearly, ternary subgroupoids of $\boldsymbol{G}$ generated by $0, 1$ and 2, respectively, are $\langle\, 0 \,\rangle = \langle\, 1 \,\rangle = \{0, 1\}$ and $\langle\, 2 \,\rangle = \{2\}$ and they are semicommutative. So, $(G, [\,])$ is a power-semicommutative ternary groupoid. It is not semicommutative, since, for instance, $[012] = 2 \neq 1 = [210]$. Also, it is not a ternary semigroup, since, for instance, $[[000]00] = [100] = 0 \neq 1 = [010] = [0[000]0]$.

In this paper we consider free objects in the class of power-semicommutative ternary groupoids.

2. Preliminaries

We will briefly mention the construction of the absolutely free ternary groupoid $\mathbb{T}_X = (T, [\,])$ over a nonempty set X, by using the construction of the absolutely free vector valued groupoid ([13]), i.e. an (n, m)-groupoid, for $n = 3$ and $m = 1$, over X, where $X \cap \{[\,]\} = \emptyset$. Let $T_0, T_1, \ldots, T_p, \ldots,$ be a sequence of disjoint sets defined by

$$T_0 = X, \qquad T_1 = \{(t_1, t_2, t_3) \mid t_1, t_2, t_3 \in T_0\},$$

$$T_{p+1} = \{(t_1, t_2, t_3) \mid t_1, t_2, t_3 \in T_0 \cup T_1 \cup \cdots \cup T_p \ \wedge \ (\exists i \in \{1, 2, 3\}) \ t_i \in T_p\}.$$

If $t \in T_p$, then we say that t has a *hierarchy* p and denote it by $\chi(t) = p$.

Put $T = \bigcup \{T_p \mid p \geq 0\}$. Define a ternary operation $[\,] : T \times T \times T \to T$

by $[t_1t_2t_3] = (t_1, t_2, t_3)$.

Then $\mathbb{T}_X$ is a ternary groupoid generated by X that has the universal mapping property in the class of ternary groupoids over X, i.e. it is the absolutely free ternary groupoid over X. The set X is the set of prime elements. Namely, an element $t \in T$ is said to be *prime* in $\mathbb{T}_X$ if $t \neq [t_1t_2t_3]$ for every $t_i \in T$, $i = 1, 2, 3$ and *non-prime* in the opposite case. For every $t \in T$ we define the *length* $\mid t \mid$ of t, the *set of subterms* $P(t)$ of t and the *set of variables* $\mathtt{var}(t)$ of t, by:

$t \in X \;\Rightarrow\; \mid t \mid = 1, \qquad t = [t_1t_2t_3] \;\Rightarrow\; \mid t \mid = \mid t_1 \mid + \mid t_2 \mid + \mid t_3 \mid;$

$t \in X \;\Rightarrow\; P(t) = \{t\}, \quad t = [t_1t_2t_3] \;\Rightarrow\; P(t) = \{t\} \cup P(t_1) \cup P(t_2) \cup P(t_3);$

$t \in X \Rightarrow \mathtt{var}(t) = \{t\}, \quad t = [t_1t_2t_3] \Rightarrow \mathtt{var}(t) = \mathtt{var}(t_1) \cup \mathtt{var}(t_2) \cup \mathtt{var}(t_3).$

A ternary groupoid $\boldsymbol{G}$ is said to be *injective* if

$$[a_1a_2a_3] = [b_1b_2b_3] \;\Rightarrow\; a_1 = b_1, a_2 = b_2, a_3 = b_3.$$

The absolutely free ternary groupoid $\mathbb{T}_X$ is injective. It is characterized by:

Theorem 1. ([16], Thm.11.2) *A ternary groupoid $\boldsymbol{G}$ is absolutely free over a nonempty set X if and only if the following two conditions are satisfied:*
  (1) *$\boldsymbol{G}$ is injective;*
  (2) *X is the set of prime elements in $\boldsymbol{G}$ that generates $\boldsymbol{G}$.*

Remark. We refer to this theorem as *Bruck Theorem* for the class of all ternary groupoids. This theorem for the binary case is stated in [5] as Lemma 1.5.

The notion of ternary groupoid powers is considered in [8] and [11]. We denote by $\mathbb{E} = (E, [\,])$ the ternary groupoid of terms over one-element set $\{e\}$. Note that

$$E = \{e, [\overset{3}{e}], [ee\overset{3}{e}], [[\overset{3}{e}]ee], [e[\overset{3}{e}]e], [e\overset{3}{e}[\overset{3}{e}]], [[\overset{3}{e}]e[\overset{3}{e}]], [[\overset{3}{e}][\overset{3}{e}]e], [ee[ee\overset{3}{e}]], \dots\}.$$

The elements of the set E are called *ternary groupoid powers* and we denote them by $f, g, h, f_i, \dots$ Every element $f \in E$ induces on a ternary groupoid $\boldsymbol{G}$ a transformation $f^{\boldsymbol{G}} : G \to G$, called the *interpretation* of f in $\boldsymbol{G}$ defined by:

$$(\forall a \in G) \quad f^{\boldsymbol{G}}(a) = \varphi_a(f),$$

where $\varphi_a : \mathbb{E} \to \boldsymbol{G}$ is the homomorphism such that $\varphi_a(e) = a$. More explicitly:

$$e(a) = a, \quad [f_1f_2f_3](a) = [f_1(a)f_2(a)f_3(a)]$$

where $f_i(a)$ stands for $f_i^{\boldsymbol{G}}(a)$, $i = 1, 2, 3$. Note that if $\boldsymbol{G} = \mathbb{E}$, then the above formulas, for every $g \in \mathbb{E}$, become

$$e(g) = g, \quad [f_1f_2f_3](g) = [f_1(g)f_2(g)f_3(g)].$$

If one defines an operation $\circ$ on E by $f \circ g = f(g)$, then one obtains an algebra $(E, [\ ], \circ)$ with one ternary and one binary operation, $[\ ]$ and $\circ$ respectively, such that:

$$e \circ g = g, \quad [f_1 f_2 f_3] \circ g = [(f_1 \circ g)(f_2 \circ g)(f_3 \circ g)],$$

for any $f_1, f_2, f_3, g \in E$. It is easily shown that $(E, \circ, e)$ is a cancellative monoid.

Proposition 1. ([8], [11]) *If $t, u \in T$ and $f, g \in E$, then:*
   (a) $|f(t)| = |f| \cdot |t|$;
   (b) $t \in P(f(t))$;
   (c) $f(t) = f(u) \Rightarrow t = u$;
   (d) $f(t) = g(t) \Rightarrow f = g$;
   (e) $f(t) = g(u) \wedge (|f| = |g| \vee |t| = |u|) \Leftrightarrow f = g \wedge t = u$;
   (f) $f(t) = g(u) \wedge |t| > |u| \Leftrightarrow (\exists! \, h \in E \setminus \{e\})(t = h(u) \wedge g = f(h))$.

Definition 1. Given a ternary groupoid $\boldsymbol{G} = (G, [\ ])$, an element $a \in G$ is said to be *primitive* in $\boldsymbol{G}$ if $a \neq f(c)$, for any $c \in G$ and any $f \in E \setminus \{e\}$; a is said to be *potent* in $\boldsymbol{G}$ if it is not primitive. Specially, primitive elements and potent elements in $\mathbb{T}_X$ are called *primitive ternary terms* and *potent ternary terms*, respectively.

Proposition 2. ([7]) *The following conditions are equivalent:*
   (a) *v is primitive in $\mathbb{T}_X$;*
   (b) *$(\forall u \in T) \, (\forall f \in E) \, (v = f(u) \Rightarrow f = e)$;*
   (c) *$(\forall u \in T) \, (\forall f \in E) \, (v = f(u) \Rightarrow v = u)$.*

The proof of the next proposition can be obtained in the same manner as in the binary case ([7]).

Proposition 3. *For any term $t \in T$ there is a unique primitive ternary term $u \in T$ and a unique $f \in E$ such that $t = f(u)$.*

In that case we call u the *base* of t, f the *power* of t, $|f|$ the *exponent* of t, and denote them by $\underline{t} = u$, $t^\sim = f$ and $|t^\sim|$, respectively.

Proposition 4. *Let t_1, t_2, t_3 be ternary terms.*
   (a) *If t_i, t_j have different bases for at least one pair (t_i, t_j), where $i, j \in \{1, 2, 3\}$, then: $\underline{[t_1 t_2 t_3]} = [t_1 t_2 t_3]$ and $([t_1 t_2 t_3])^\sim = e$.*
   (b) *If $\underline{t_1} = \underline{t_2} = \underline{t_3} = u$, then $\underline{[t_1 t_2 t_3]} = u$ and $([t_1 t_2 t_3])^\sim = [t_1^\sim t_2^\sim t_3^\sim]$.*

Proof. (a) Let $t_1, t_2, t_3 \in T$. By Prop.3, let $t_i = f_i(u_i)$, where $u_i = \underline{t_i}$, $f_i = t_i^\sim$ for $i \in \{1, 2, 3\}$ and let $t = [t_1 t_2 t_3] = f(u)$, where $u = \underline{t}$, $f = t^\sim$.

Without loss of generality, let $u_1 \neq u_2$. We claim that $f = e$. Suppose that $f = [f_1 f_2 f_3]$. Then $f(u) = [f_1 f_2 f_3](u) = [f_1(u) f_2(u) f_3(u)]$ and $f(u) =$

$[t_1 t_2 t_3] = [f_1(u_1) f_2(u_2) f_3(u_3)]$. By the injectivity of $\mathbb{T}_X$, $f_i(u_i) = f_i(u)$ and by Prop. 1 (c) it follows that $u_i = u$, $i \in \{1, 2, 3\}$, and this contradicts the supposition $u_1 \neq u_2$. Thus, $f = e$, i.e. $\underline{t} = u = [t_1 t_2 t_3]$.

(b) Let $\underline{t_1} = \underline{t_2} = \underline{t_3} = u$, $t_i = f_i(u)$, $i \in \{1, 2, 3\}$ and $t = [t_1 t_2 t_3] = g(v)$. Then $g(v) = [f_1(u) f_2(u) f_3(u)]$. Since v is a primitive term and $\|[f_1(u) f_2(u) f_3(u)]\| \geq 3$, it follows that $g \neq e$. Let $g = [g_1 g_2 g_3]$. Then

$$[f_1(u) f_2(u) f_3(u)] = g(v) = [g_1 g_2 g_3](v) = [g_1(v) g_2(v) g_3(v)].$$

By the injectivity of $\mathbb{T}_X$ it follows that $g_i(v) = f_i(u)$, $i \in \{1, 2, 3\}$. As u and v are primitive terms, neither $|u| > |v|$ nor $|v| > |u|$ (Prop.1 (f)). Therefore $|u| = |v|$. By Prop.1 (e) it follows that $f_i = g_i$, $i \in \{1, 2, 3\}$ and $u = v$. Thus, $\underline{[t_1 t_2 t_3]} = u$. By $[t_1 t_2 t_3] = [f_1(u) f_2(u) f_3(u)]$ it follows that

$$([t_1 t_2 t_3])^\smallsmile = g = [f_1(u) f_2(u) f_3(u)] = [t_1^\smile t_2^\smile t_3^\smile].$$

$\square$

3. A construction of canonical objects in $\mathcal{P}_{sc}$

The ternary subgroupoid of a ternary groupoid $\boldsymbol{G} = (G, [\,])$ generated by an element $a \in G$ will be denoted by $\langle\, a\, \rangle$ and called the *mono-generated ternary subgroupoid* of $\boldsymbol{G}$ with the generator a. A ternary groupoid $\boldsymbol{G}$ is said to be *power-semicommutative* if every mono-generated ternary subgroupoid of $\boldsymbol{G}$ is semicommutative. We denote the class of power-semicommutative ternary groupoids by $\mathcal{P}_{sc}$. Mono-generated ternary subgroupoids of a ternary groupoid can be characterized as:

$$(\forall a \in G) \langle\, a\, \rangle = \{f(a) : f \in E\}.$$

This enables to obtain an axiom system for $\mathcal{P}_{sc}$.

Proposition 5. *A ternary groupoid $\boldsymbol{G}$ is power-semicommutative if and only if*

$$(\forall a \in G) \, (\forall f_1, f_2, f_3 \in E) \, [f_1(a) f_2(a) f_3(a)] = [f_3(a) f_2(a) f_1(a)].$$

As a consequence of Proposition 5 it follows that the class of power-semicommutative ternary groupoids $\mathcal{P}_{sc}$ is a variety of ternary groupoids defined by the system of identities

$$\{[f_1(x) f_2(x) f_3(x)] \approx [f_3(x) f_2(x) f_1(x)] : \ f_1, f_2, f_3 \in E\}.$$

We are looking for a canonical groupoid in $\mathcal{P}_{sc}$.

Definition 2. A groupoid $\boldsymbol{R} = (R, [\ ]^*)$ is called a *canonical power-semicommutative ternary groupoid* if the following conditions are satisfied:

(C0) $X \subseteq R \subseteq T$ and $t \in R \Rightarrow P(t) \subseteq R$;

(C1) $t_1, t_2, t_3, [t_1t_2t_3] \in R \Rightarrow [t_1t_2t_3]^* = [t_1t_2t_3]$;

(C2) $\boldsymbol{R}$ is a free ternary groupoid in $\mathcal{P}_{sc}$ over X (i.e. $\mathcal{P}_{sc}$-free over X).

Further on we will use an ordering on T assuming that X is a linearly ordered set ([9]), where the ordering relation is denoted by $\leq$. Ternary terms are ordered so that a ternary term of a particular hierarchy comes before any term of a higher hierarchy and among ternary terms of equal hierarchy lexicographical ordering is used, with ternary terms earlier in the lexicographical ordering coming first. The precise definition is given below.

Definition 3. Let X be a linearly ordered set and $t, u \in T$.

  (i) $t, u \in X \Rightarrow (t \leq u \text{ in } T \Leftrightarrow t \leq u \text{ in } X)$;

  (ii) $\chi(t) < \chi(u) \Rightarrow t < u$;

  (iii) If $\chi(t) = \chi(u) \geq 1$ and $t \neq u$, where $t = [t_1t_2t_3]$, $u = [u_1u_2u_3]$, then $t < u$ if and only if
  $$[t_1 < u_1 \lor (t_1 = u_1 \land t_2 < u_2) \lor (t_1 = u_1 \land t_2 = u_2 \land t_3 < u_3)].$$

Then $\leq$ is a linear ordering of T.

We will use the free semicommutative ternary groupoid constructed as follows ([9]). Define a subset T_{sc} of T by
$$T_{sc} = X \cup \{t \in T \setminus X : [t_1t_2t_3] \in P(t) \Rightarrow t_1 \leq t_3\}$$
and an operation $[\,]'$ on T_{sc} by $[t_1t_2t_3]' = [t_1t_2t_3]$ if $t_1 \leq t_3$, and, $[t_1t_2t_3]' = [t_3t_2t_1]$ if $t_3 < t_1$, for every $t_1, t_2, t_3 \in T_{sc}$. Then $\mathbb{T}_{sc} = (T_{sc}, [\,]')$ is a canonical semicommutative ternary groupoid over X.

Specially, $\mathbb{E}_{sc} = (E_{sc}, [\,]')$ is a canonical semicommutative ternary groupoid over e, where $E_{sc} = \{e\} \cup \{f \in E \setminus \{e\} : [f_1f_2f_3] \in P(f) \Rightarrow f_1 \leq f_3\}$.

Let $\psi : E \to E_{sc}$ be a homomorphism from $\mathbb{E}$ into $\mathbb{E}_{sc}$ such that $\psi(e) = e$. For every $f \in E$ put $\psi(f) = \overline{f}$. If $f = [f_1f_2f_3] \in E$, we obtain that $\overline{[f_1f_2f_3]} = [\overline{f_1}\ \overline{f_2}\ \overline{f_3}]'$. For instance, if $f = [\overset{3}{e}]$, then $\psi(f) = \overline{[\overset{3}{e}]} = [\overset{3}{e}]$, but if $f = [[\overset{3}{e}]ee]$, then $\psi(f) = \overline{[[\overset{3}{e}]ee]} = [\overline{[\overset{3}{e}]}\ \overline{e}\ \overline{e}]' = [[\overset{3}{e}]ee]' = [ee[\overset{3}{e}]]$.

Clearly, if $f \in E_{sc}$, then $f = \overline{f}$.

A term $t \in T$ is said to be

  (1) *laterally order-regular* if $t \in X$ or $(t = [t_1t_2t_3] \land t_1 \leq t_3)$.
      Specially, a ternary groupoid power f is laterally order-regular if and only if $f = e$ or $(f = [f_1f_2f_3] \land f_1 \leq f_3)$.

  (2) *order-regular* if $t \in X$ or $(t = [t_1t_2t_3] \land t_1 \leq t_2 \leq t_3)$.
      Specially, a ternary groupoid power f is order-regular if and only if $f = e$ or $(f = [f_1f_2f_3] \land f_1 \leq f_2 \leq f_3)$.

By induction on the length of f, one can show the following two statements as in the binary case ([6]).

Lemma 1. *Let $f, g, h \in E$.*

(a) *If $f < g$, then $f(h) < g(h)$.*

(b) *$f(h)$ is laterally order-regular (order-regular) if and only if f and h are laterally order-regular (order-regular).*

To obtain a description of the canonical objects in $\mathcal{P}_{sc}$, define the set

$$R = \{t \in T : u \in P(t) \Rightarrow u^\sim \in E_{sc}\}. \tag{1}$$

In other words, a term $t \in R$ if and only if the power of any subterm u of t is laterally order-regular, i.e. if $u \in X$ or $(u = [u_1 u_2 u_3] \Rightarrow u_1^\sim \leq u_3^\sim)$.

From (1) we obtain the following consequences:

(a) $X \subset R \subset T$;

(b) $[t_1 t_2 t_3] \in R \Rightarrow t_1, t_2, t_3 \in R$;

(c) $t_1, t_2, t_3 \in R \Rightarrow \{[t_1 t_2 t_3] \notin R \Leftrightarrow \underline{t_1} = \underline{t_2} = \underline{t_3} \wedge t_3^\sim < t_1^\sim\}$;

(d) $t_1, t_2, t_3 \in T \Rightarrow \{[t_1 t_2 t_3] \in R \Leftrightarrow t_1, t_2, t_3 \in R \wedge$
$$\wedge ((\exists i, j \in \{1, 2, 3\}) \ \underline{t_i} \neq \underline{t_j} \ \vee \ (\underline{t_1} = \underline{t_2} = \underline{t_3} \ \wedge \ t_1^\sim \leq t_3^\sim))\}.$$

Define an operation $[\ ]^*$ on R by:

$$t_1, t_2, t_3 \in R \ \Rightarrow \ [t_1 t_2 t_3]^* = \begin{cases} [t_1 t_2 t_3], & \text{if } [t_1 t_2 t_3] \in R \\ [t_3 t_2 t_1], & \text{if } \underline{t_1} = \underline{t_2} = \underline{t_3} \ \wedge \ t_3^\sim < t_1^\sim. \end{cases} \tag{2}$$

Theorem 2. *The groupoid $\boldsymbol{R} = (R, [\ ]^*)$ is a canonical power-semicommutative ternary groupoid over X.*

Proof. From (c), (d) and (2) it follows that $\boldsymbol{R} = (R, [\ ]^*)$ is a ternary groupoid. It is clear that X is the set of primes in $\boldsymbol{R}$ and generates $\boldsymbol{R}$. To prove that $\boldsymbol{R} \in \mathcal{P}_{sc}$, it suffices to show that every mono-generated ternary subgroupoid of $\boldsymbol{R}$ is semicommutative. For that reason we will use the interpretation of $f \in E$ in $\boldsymbol{R}$, denoted by $f_*(t)$:

$$(\forall t \in R) \ (e_*(t) = t \ \wedge \ [f_1 f_2 f_3]_*(t) = [f_{1*}(t) f_{2*}(t) f_{3*}(t)]^*) \tag{3}$$

By induction on the length of f ($|f| = 2k - 1$, $k \in \mathbb{N}$) and using Lemma 1 (a), one can show that for every $t \in R$:

1) $f \in E_{sc} \Rightarrow f_*(t) = f(t)$;   2) $f \in E \Rightarrow f_*(t) = \overline{f}(t)$.

Namely, for 1): if $k = 1$, then $f = e$ and $e_*(t) = t = e(t)$. Assume that the claim is true for all the powers f up to some length $2k - 1$ and let $f = [f_1 f_2 f_3] \in E_{sc}$ be such that $|f| = 2k + 1$. Then $|f_i| \leq 2k - 1$, $i \in \{1, 2, 3\}$ and thus

$$f_*(t) = [f_1 f_2 f_3]_*(t) = [f_{1*}(t) f_{2*}(t) f_{3*}(t)]^* =$$
$$= [f_1(t) f_2(t) f_3(t)]^* = [f_1 f_2 f_3]'(t) = f(t).$$

For 2): let $f \in E$. If $f \in E_{sc}$, then by (a), $f_*(t) = f(t) = \overline{f}(t)$. Therefore, let $f \in E \setminus E_{sc}$. For the shortest $f \in E \setminus E_{sc}$, i.e. for $f = [[\overset{3}{e}]ee]$, we have

$$f_*(t) = [[\overset{3}{e}]ee]_*(t) = [[\overset{3}{e}]_*(t)e_*(t)e_*(t)]^* =$$
$$= [[\overset{3}{e}](t)e(t)e(t)]^* = [[\overset{3}{e}]ee]'(t) = [ee[\overset{3}{e}]](t) = \overline{f}(t).$$

Assume that the claim is true up to some f with $|f| = 2k - 1$ and let $f = [f_1 f_2 f_3] \in E \setminus E_{sc}$ be with $|f| = 2k + 1$. Then $|f_i| \leq 2k - 1$, $i \in \{1, 2, 3\}$ and thus

$$f_*(t) = [f_1 f_2 f_3]_*(t) = [f_{1*}(t)f_{2*}(t)f_{3*}(t)]^* =$$
$$= [\overline{f_1}(t)\overline{f_2}(t)\overline{f_3}(t)]^* = [\overline{f_1}\ \overline{f_2}\ \overline{f_3}]'(t) = \overline{[f_1 f_2 f_3]}(t) = \overline{f}(t).$$

Now we will show that every mono-generated ternary subgroupoid $\langle\, t\, \rangle_*$, $t \in R$, is a semicommutative ternary subgroupoid of $\boldsymbol{R}$, i.e. $\boldsymbol{R} \in \mathcal{P}_{sc}$.

Namely, let $t \in R$. Then by 2), $f_*(t) = \overline{f}(t)$ for every $f \in E$, i.e. $\{f_*(t) : f \in E\} \subseteq \{f_*(t) : f \in E_{sc}\}$. Since $E_{sc} \subset E$ and by 1), it follows that the inverse inclusion is true as well. Therefore, $\{f_*(t) : f \in E\} = \{f_*(t) : f \in E_{sc}\}$. Using the fact that for any $t \in R$, $\langle\, t\, \rangle_* = \{f_*(t) : f \in E\}$ one obtains that $\langle\, t\, \rangle_* \in \mathcal{S}_c$.

It remains to show that the ternary groupoid $\boldsymbol{R}$ has the universal mapping property for $\mathcal{P}_{sc}$ over X.

Let $\boldsymbol{G} = (G, [\,]'') \in \mathcal{P}_{sc}$, $\lambda : X \to G$ be any mapping and $\varphi : \mathbb{T}_X \to \boldsymbol{G}$ be the homomorphism that is an extension of λ. If $t_1, t_2, t_3 \in R$ and $[t_1 t_2 t_3] \in R$, then $\varphi([t_1 t_2 t_3]^*) = \varphi([t_1 t_2 t_3]) = [\varphi(t_1)\varphi(t_2)\varphi(t_3)]''$. If $[t_1 t_2 t_3] \notin R$, then $\underline{t_1} = \underline{t_2} = \underline{t_3}$ and $t_3^{\frown} < t_1^{\frown}$. Let $\underline{t_i} = u$ and $t_i = f_i(u)$ for some $f_i \in E$, $i \in \{1, 2, 3\}$, i.e. $t_i \in \langle\, u\, \rangle_*$. It is clear that $\varphi(t_i) \in \langle\, \varphi(u)\, \rangle$ and so

$$\varphi([t_1 t_2 t_3]^*) = \varphi([t_1 t_2 t_3]) = [\varphi(t_3)\varphi(t_2)\varphi(t_1)]'' = [\varphi(t_1)\varphi(t_2)\varphi(t_3)]'',$$

since $\boldsymbol{G} \in \mathcal{P}_{sc}$. Therefore, $\varphi|_R : \boldsymbol{R} \to \boldsymbol{G}$ is a homomorphism that extends λ. Hence, $\boldsymbol{R}$ is $\mathcal{P}_{sc}$-free groupoid over X that has the properties (C0) and (C1). Thus, (1) and (2) give a description of the canonical objects in the variety of power-semicommutative ternary groupoids. $\qquad \square$

4. A CHARACTERIZATION OF $\mathcal{P}_{sc}$-FREE TERNARY GROUPOIDS

We will define a subclass of the class $\mathcal{P}_{sc}$ that is larger than the class of $\mathcal{P}_{sc}$-free ternary groupoids called the class of $\mathcal{P}_{sc}$-injective ternary groupoids in order to obtain a characterization of free objects in $\mathcal{P}_{sc}$. For that reason we will use the properties of the corresponding $\mathcal{P}_{sc}$-canonical ternary groupoid $\boldsymbol{R} = (R, [\,]^*)$ related to the elements of R that are not prime in $\boldsymbol{R}$. The class of $\mathcal{P}_{sc}$-injective ternary groupoids will be successfully defined if the following two conditions are satisfied:

1) the class of $\mathcal{P}_{sc}$-injective ternary groupoids should enable a characterization of $\mathcal{P}_{sc}$-free ternary groupoids analogously as in Theorem 1;

2) the class of $\mathcal{P}_{sc}$-free ternary groupoids has to be a proper subclass of the class of $\mathcal{P}_{sc}$-injective ternary groupoids.

Therefore, we will state some properties that are related to the non-prime elements in $(R, [\ ]^*)$ that are necessary for defining the notion of $\mathcal{P}_{sc}$-injectivity.

Note that $t \in R$ is primitive in $\boldsymbol{R}$ if and only if t is primitive in $\mathbb{T}_X$.

Similarly as in Prop.3 and Prop.4 one obtains:

Lemma 2. *Let* $t_1, t_2, t_3 \in R$.

   a) *For any* $t \in R$ *there is a unique primitive term* $u \in R$ *and a unique* $f \in E_{sc}$, *such that* $t = f_*(u)$. *Here we also write,* $\underline{t} = u$ *and* $t^\sim = f$.
   b) *If* t_i, t_j *have different bases for at least one pair* t_i, t_j, $i, j \in \{1, 2, 3\}$, *then:* $\underline{[t_1 t_2 t_3]} = [t_1 t_2 t_3]$ *and* $([t_1 t_2 t_3])^\sim = e$.
   c) *If* $\underline{t_1} = \underline{t_2} = \underline{t_3} = u$, *then:* $\underline{[t_1 t_2 t_3]} = u$ *and* $([t_1 t_2 t_3])^\sim = [t_1^\sim t_2^\sim t_3^\sim]'$.

If a term $t \in R$ is not prime in $\boldsymbol{R}$, then $t = [t_1 t_2 t_3]^*$ for $t_1, t_2, t_3 \in R$. In that case we say that (t_1, t_2, t_3) is a *triple of divisors* of t in $\boldsymbol{R}$, where t_1 is the *left*, t_2 is the *middle* and t_3 is the *right* divisor of t; we write $(t_1, t_2, t_3)|t$.

Proposition 6. *Let* t *be a non-prime element in* $\boldsymbol{R}$. *Then there is a unique triple* (t_1, t_2, t_3) *of* R^3 *such that* $t = [t_1 t_2 t_3]^* = [t_1 t_2 t_3]$.

We call $[t_1 t_2 t_3]$ the *canonical triple of divisors* of t in $\boldsymbol{R}$. This triple coincides with a triple of divisors in $\mathbb{T}_X$.

Proof. If $t \in R \setminus X$ then $t \in T \setminus X$ and thus $t = [t_1 t_2 t_3]$ for a unique triple (t_1, t_2, t_3) of T^3. Since $[t_1 t_2 t_3] \in R$ it follows that $t_1, t_2, t_3 \in R$ and thus $t = [t_1 t_2 t_3]^* = [t_1 t_2 t_3]$. $\qquad\qquad\square$

Proposition 7. *If* $u_i, v_i \in R$, $i \in \{1, 2, 3\}$, *are such that* $[u_1 u_2 u_3]^* = [v_1 v_2 v_3]^*$, *and* $[u_1 u_2 u_3] \notin R$ *or* $[v_1 v_2 v_3] \notin R$, *then* $\underline{u_i} = \underline{v_i}$, $i \in \{1, 2, 3\}$, *and* $u_2^\sim = v_2^\sim$, $\{u_1^\sim, u_3^\sim\} = \{v_1^\sim, v_3^\sim\}$.

Proof. If $[u_1 u_2 u_3] \notin R$ and $[v_1 v_2 v_3] \in R$, then by the consequence (c) of (1) (Definition of R), $\underline{u_1} = \underline{u_2} = \underline{u_3} = \alpha$, $u_3^\sim < u_1^\sim$, where α is a primitive term in $\boldsymbol{R}$, and thus there are $f_i \in E_{sc}$ such that $u_i = f_i(\alpha)$, where $f_i = u_i^\sim$, $i \in \{1, 2, 3\}$. Therefore, the equation $[u_1 u_2 u_3]^* = [v_1 v_2 v_3]^*$ becomes $[u_3 u_2 u_1] = [v_1 v_2 v_3]$ in $\mathbb{T}_X$, i.e. $[f_3(\alpha) f_2(\alpha) f_1(\alpha)] = [v_1 v_2 v_3]$, that implies $f_3(\alpha) = v_1$, $f_2(\alpha) = v_2$, $f_1(\alpha) = v_3$ and $\underline{v_1} = \underline{v_2} = \underline{v_3} = \alpha$, $v_1^\sim = f_3$, $v_2^\sim = f_2$, $v_3^\sim = f_1$. Thus, $\underline{u_i} = \underline{v_i}$, $i \in \{1, 2, 3\}$, and, $u_1^\sim = v_3^\sim$, $u_2^\sim = v_2^\sim$, $u_3^\sim = v_1^\sim$. The same conclusion is true when $[u_1 u_2 u_3] \in R$ and $[v_1 v_2 v_3] \notin R$.

If $[u_1 u_2 u_3] \notin R$ and $[v_1 v_2 v_3] \notin R$, then $\underline{u_1} = \underline{u_2} = \underline{u_3} = \alpha$, $u_3^\sim < u_1^\sim$,

where α is a primitive element in $\boldsymbol{R}$ and $\underline{v_1} = \underline{v_2} = \underline{v_3} = \beta$, $v_3^{\sim} < v_1^{\sim}$, where β is a primitive element in $\boldsymbol{R}$. So, we can write $u_i = u_i^{\sim}(\alpha)$, $v_i = v_i^{\sim}(\beta)$, $i \in \{1, 2, 3\}$. The equation $[u_1 u_2 u_3]^* = [v_1 v_2 v_3]^*$ becomes $[u_3 u_2 u_1] = [v_3 v_2 v_1]$ in $\mathbb{T}_X$ and thus $u_i = v_i$, i.e. $u_i^{\sim}(\alpha) = v_i^{\sim}(\beta)$. This implies that $u_i^{\sim} = v_i^{\sim}$, $\alpha = \beta$, i.e. $\underline{u_i} = \underline{v_i}$, $i \in \{1, 2, 3\}$ and $u_1^{\sim} = v_1^{\sim}$, $u_3^{\sim} = v_3^{\sim}$.

Therefore, $[u_1 u_2 u_3]^* = [v_1 v_2 v_3]^*$ and $([u_1 u_2 u_3] \notin R$ or $[v_1 v_2 v_3] \notin R)$ implies that $\underline{u_i} = \underline{v_i}$, $i \in \{1, 2, 3\}$, and $u_2^{\sim} = v_2^{\sim}$, $\{u_1^{\sim}, u_3^{\sim}\} = \{v_1^{\sim}, v_3^{\sim}\}$. $\square$

By Prop.6 and Prop.7 one obtains the following statement.

If $[u_1 u_2 u_3]^* = [v_1 v_2 v_3]^*$, then:

1) $u_1 = v_1$, $u_2 = v_2$, $u_3 = v_3$, when $[u_1 u_2 u_3], [v_1 v_2 v_3] \in R$ or $[u_1 u_2 u_3]$, $[v_1 v_2 v_3] \notin R$;

2) $u_1 = v_3$, $u_3 = v_1$, when $([u_1 u_2 u_3] \notin R$ and $[v_1 v_2 v_3] \in R)$ or $([u_1 u_2 u_3] \in R$ and $[v_1 v_2 v_3] \notin R)$.

In that case $\underline{u_i} = \underline{v_i}$ and $u_i^{\sim} = v_i^{\sim}$, for $i = 1, 2, 3$.

The pertinent result in this regard is the following

Corollary 1. *Let $w \in R$ be a non-prime element in $\boldsymbol{R}$, i.e. $w = [v_1 v_2 v_3]^*$.*

a) If w is primitive in $\boldsymbol{R}$, then (v_1, v_2, v_3) is the unique triple of divisors of w (the canonical one).

b) If w is a potent term in $\boldsymbol{R}$, i.e. $\underline{v_1} = \underline{v_2} = \underline{v_3}$, then (v_3, v_2, v_1) is the unique triple of divisors in $\boldsymbol{R}$ when $v_1^{\sim} = v_3^{\sim}$, and (v_1, v_2, v_3), (v_3, v_2, v_1) are the only two triples of divisors of w when $v_1^{\sim} > v_3^{\sim}$.

Now we will introduce the concept of injectivity in $\mathcal{P}_{sc}$.

Definition 4. A ternary groupoid $\boldsymbol{H} = (H, [\,])$ is said to be $\mathcal{P}_{sc}$-injective if the following conditions are satisfied.

$(I0)$ $\boldsymbol{H} \in \mathcal{P}_{sc}$.

$(I1)$ For any $a \in H$ there is a unique primitive element $b \in H$ and a unique $f \in E_{sc}$ such that $a = f(b)$.

(Then b is called the *base* of a and f the *power* of a; we write: $\underline{a} = b$, $a^{\sim} = f$, respectively).

$(I2)$ Let $a_1, a_2, a_3 \in H$. If $\underline{a_i} \neq \underline{a_j}$ for at least one pair (a_i, a_j), $i, j \in \{1, 2, 3\}$, then $\underline{[a_1 a_2 a_3]} = [a_1 a_2 a_3]$ and $([a_1 a_2 a_3])^{\sim} = e$.

If $\underline{a_i} = b$, $i = \overline{1, 2, 3}$, then $\underline{[a_1 a_2 a_3]} = b$ and $([a_1 a_2 a_3])^{\sim} = [a_1^{\sim} a_2^{\sim} a_3^{\sim}]$ when $a_1^{\sim} \leq a_3^{\sim}$ or $([a_1 a_2 a_3])^{\sim} = [a_3^{\sim} a_2^{\sim} a_1^{\sim}]$ when $a_3^{\sim} \leq a_1^{\sim}$.

$(I3)$ Let $a \in H$ be a non-prime element in $\boldsymbol{H}$, i.e. $a = [a_1 a_2 a_3]$.

If a is primitive in $\boldsymbol{H}$, then (a_1, a_2, a_3) is the unique triple of divisors of a.

If a is potent in $\boldsymbol{H}$, i.e. $\underline{a_1} = \underline{a_2} = \underline{a_3}$, then:

a) (a_1, a_2, a_3) is the unique triple of divisors of a when $a_1^{\sim} = a_3^{\sim}$;

b) (a_1, a_2, a_3), (a_3, a_2, a_1) are the unique two triples of divisors when $a_{\widetilde{1}} > a_{\widetilde{3}}$.

By Theorem 2, Lemma 2 and Cor.1, the ternary groupoid $\boldsymbol{R}$ satisfies the conditions $(I0) - (I3)$. Since any $\mathcal{P}_{sc}$-free ternary groupoid over X is isomorphic to $\boldsymbol{R}$, one obtains that:

Proposition 8. *Every $\mathcal{P}_{sc}$-free ternary groupoid is $\mathcal{P}_{sc}$-injective.*

In order to prove Bruck theorem for the class $\mathcal{P}_{sc}$ we will use the following Lemma 3 (that can be proved in a similar way as in [9], Lemma 3.1).

Lemma 3. *Let $\boldsymbol{H} = (H, [\,])$ be a $\mathcal{P}_{sc}$-injective ternary groupoid such that the set P of prime elements in $\boldsymbol{H}$ is nonempty and generates $\boldsymbol{H}$. If $C_0 = P$, $C_1 = [C_0 C_0 C_0]$, $C_{k+1} = \{a \in H \setminus P : (d_1, d_2, d_3)|a \Rightarrow$*
$$\Rightarrow \{d_1, d_2, d_3\} \subset C_0 \cup \cdots \cup C_k \wedge \{d_1, d_2, d_3\} \cap C_k \neq \varnothing\},$$
then $H = \bigcup\{C_k : k \geq 0\}$, where $C_k \neq \varnothing$ for any $k \geq 0$ and $C_i \cap C_j = \varnothing$ for $i \neq j$.

Theorem 3. (Bruck theorem for $\mathcal{P}_{sc}$) *A power-semicommutative ternary groupoid $\boldsymbol{H} = (H, [\,])$ is $\mathcal{P}_{sc}$-free if and only if*
(i) $\boldsymbol{H}$ is $\mathcal{P}_{sc}$-injective
(ii) The set P of prime elements in $\boldsymbol{H}$ is nonempty and generates $\boldsymbol{H}$.
Then the set of primes in $\boldsymbol{H}$ is the unique $\mathcal{P}_{sc}$-free generating set of $\boldsymbol{H}$.

Proof. If $\boldsymbol{H}$ is $\mathcal{P}_{sc}$-free, then by Prop.8 and the fact that $\boldsymbol{H}$ is isomorphic to $\boldsymbol{R}$ it follows that (i) and (ii) are fulfilled.

Conversely, let (i) and (ii) hold. By Lemma 3, $H = \bigcup\{C_k : k \geq 0\}$. To show that $\boldsymbol{H}$ is $\mathcal{P}_{sc}$-free over the set P, we need to show that it has the universal mapping property for $\mathcal{P}_{sc}$ over P. For this reason let $\boldsymbol{G} = (G, [\,]')$ be a power-semicommutative ternary groupoid and $\lambda : P \to G$ be a mapping. Define a sequence of mappings $\varphi_k : C_k \to G$ inductively by: $\varphi_0 = \lambda$ and let $\varphi_i : C_i \to G$ be defined for every $i \leq k$. If $a \in C_{k+1}$ and (d_1, d_2, d_3) is a triple of divisors of a where $d_1 \in C_p$, $d_2 \in C_q$, $d_3 \in C_r$, then $p, q, r \leq k$. Putting

$$\varphi_{k+1}([d_1 d_2 d_3]) = [\varphi_k(d_1) \varphi_k(d_2) \varphi_k(d_3)]'$$

we obtain that $\varphi = \cup\{\varphi_i : i \geq 0\}$ is a mapping from H into G that is a homomorphism. Therefore:

$$\varphi(a) = [\varphi_k(d_1) \varphi_k(d_2) \varphi_k(d_3)]' = [\varphi(d_1) \varphi(d_2) \varphi(d_3)]'.$$

Since $\boldsymbol{H}$ is $\mathcal{P}_{sc}$-injective it follows that (d_3, d_2, d_1) is a triple of divisors of a as well and that

$$\varphi([d_3 d_2 d_1]) = [\varphi(d_3) \varphi(d_2) \varphi(d_1)]' = (\boldsymbol{G} \in \mathcal{P}_{sc}) =$$

$$= [\varphi(d_1)\varphi(d_2)\varphi(d_3)]' = \varphi([d_1 d_2 d_3]).$$

Thus φ is a well defined mapping and a homomorphism. Hence, $\boldsymbol{H}$ is $\mathcal{P}_{sc}$-free over P. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The next example shows that there is a $\mathcal{P}_{sc}$-injective ternary groupoid that is not free in $\mathcal{P}_{sc}$, i.e. the class of free ternary groupoids in $\mathcal{P}_{sc}$ is a proper subclass of the class of $\mathcal{P}_{sc}$-injective ternary groupoids.

Example. Let X be an infinite countable set and $\boldsymbol{R} = (R, [\ ]^*)$ be the canonical power-semicommutative ternary goupoid over X. Define two subsets $H \subset R$ and $D \subset H^3$ as follows:

$H = \{t \in R : |\mathsf{var}(t)| = 1\}$,

$D = \{(t_1, t_2, t_3) \in H^3 : \mathsf{var}(t_i) \neq \mathsf{var}(t_j)$ for at least one pair of $i, j \in \{1, 2, 3\}\}$

and define a relation θ in D by:

$$(t_1, t_2, t_3)\theta(u_1, u_2, u_3) \Leftrightarrow t_2 = u_2 \wedge \{t_1, t_3\} = \{u_1, u_3\}.$$

It is clear that θ is an equivalence relation in D. If $\mathsf{var}(t_1) \neq \mathsf{var}(t_3)$, then the equivalence class $(t_1, t_2, t_3)^\theta = \{(t_1, t_2, t_3), (t_3, t_2, t_1)\}$ and if $\mathsf{var}(t_1) = \mathsf{var}(t_3)$, then $(t_1, t_2, t_3)^\theta = \{(t_1, t_2, t_3)\}$.

The set of θ-equivalence classes in D is denoted by D^θ. It is of the same cardinality as the set X and thus there is a bijective mapping $\psi : D^\theta \to X$. Define a ternary operation $[\ ]'$ on H by: for any $t_1, t_2, t_3 \in H$,

$$[t_1 t_2 t_3]' = \begin{cases} [t_1 t_2 t_3]^*, \text{ if } \mathsf{var}(t_1) = \mathsf{var}(t_2) = \mathsf{var}(t_3) \\ \psi((t_1, t_2, t_3)^\theta), \text{ if } \mathsf{var}(t_i) \neq \mathsf{var}(t_j), \text{ for some pair } i, j \in \{1, 2, 3\}. \end{cases}$$
$$(4)$$

It is clear by (4) that $\boldsymbol{H} = (H, [\ ]')$ is a ternary groupoid and $\boldsymbol{H} \in \mathcal{P}_{sc}$. Therefore, the condition $(I0)$ of Def.4 is satisfied. The conditions $(I1)$ and $(I2)$ of Def.4 are fulfilled by Lemma 2 a) and b), c) respectively.

Let $t \in H \setminus X$, i.e. $t = [t_1 t_2 t_3]'$. (The element t is not primitive in $\boldsymbol{H}$, because if it would be, we would obtain that $\mathsf{var}(t_i) \neq \mathsf{var}(t_j)$, for some pair $i, j \in \{1, 2, 3\}$ and thus $t = \psi((t_1, t_2, t_3)^\theta) \in X$, a contradiction.) Then t is a potent element in $\boldsymbol{H}$ and thus $\underline{t_1 = t_2 = t_3}$, i.e. $\mathsf{var}(t_1) = \mathsf{var}(t_2) = \mathsf{var}(t_3)$ and $t = [t_1 t_2 t_3]^*$. If $t_1^\sim = t_3^\sim$, then $t_1 = t_3$ and so $t = [t_1 t_2 t_1]'$, i.e. (t_1, t_2, t_1) is the unique pair of divisors of t in $\boldsymbol{H}$. If $t_1^\sim \neq t_3^\sim$, then $t = [t_1 t_2 t_3]' = [t_1 t_2 t_3]^* = [t_3 t_2 t_1]^* = [t_3 t_2 t_1]'$, since $[t_1 t_2 t_3]'$, $[t_3 t_2 t_1]' \in \langle t \rangle_* \in \mathcal{P}_{sc}$. Hence, the condition $(I3)$ from Def.4 is satisfied. Thus the ternary groupoid $\boldsymbol{H}$ is $\mathcal{P}_{sc}$-injective.

Since $\psi : D^\theta \to X$ is a bijective mapping, it follows that the set $X \setminus im\psi = \varnothing$, i.e. the set of primes in $(H, [\ ]')$ is empty. By Bruck theorem for $\mathcal{P}_{sc}$ it follows that the ternary groupoid $(H, [\ ]')$ is not $\mathcal{P}_{sc}$-free. Thus we have proved the following

Proposition 9. *The class of $\mathcal{P}_{sc}$-free ternary groupoids is a proper subclass of the class of $\mathcal{P}_{sc}$-injective ternary groupoids.*

Hence, the characterization of $\mathcal{P}_{sc}$-free objects by means of $\mathcal{P}_{sc}$-injective ternary groupoids is completed.

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