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On free medial semigroups

Vesna Celakoska-Jordanova
Faculty of Natural Sciences and Mathematics,
Ss. Cyril and Methodius University, Skopje
Republic of Macedonia

Abstract. *Descriptions of free objects in the variety of medial semigroups are obtained.*
Key words: *medial semigroup, free semigroup, quotient semigroup, V-canonical semigroup.*

A semigroup ***S*** is said to be ***medial*** if $abcd = acbd$, for every a, b, c, d in S. The class of all medial semigroups is a variety of semigroups defined by the identity $xyuv \approx xuyv$. We denote this variety by *Med*. If ***S*** is a medial semigroup, then the following equalities hold in ***S***:

$$px_1x_2x_3q = py_1y_2y_3q,$$

for any $p, x_1, x_2, x_3, q \in S$ and any permutation $y_1y_2y_3$ of $x_1x_2x_3$.

A free medial semigroup as a quotient structure of the free semigroup over an alphabet is constructed below. Recall that the free semigroup over a nonempty set A, is the set A^+ of all finite sequences $(a_1,a_2,\ldots,a_n)$ of elements of A endowed by the operation concatenation of sequences:

$$(a_1,a_2,\ldots,a_n)(b_1,b_2,\ldots,b_m) = (a_1,a_2,\ldots,a_n,b_1,b_2,\ldots,b_m).$$

Identifying each sequence of the form (a) with $a \in A$, we have $(a_1,a_2,\ldots,a_n) = a_1a_2\ldots a_n$. The elements of A are ***letters*** and elements of A^+ are ***words*** on A. Two words $u = a_1a_2\ldots a_n$ and $v = b_1b_2\ldots b_n$ are ***equal*** if and only if $n = m$ and $a_i = b_i$, for $i = 1,2,\ldots,n$. The number n is called the ***length*** of the word $w = a_1a_2\ldots a_n$ and it is denoted by $|w|$. For any word w we define the ***content*** cnt(w) inductively by:

$$a \in A \Rightarrow \text{cnt}(w) = \{a\},\ \ w = uv \Rightarrow \text{cnt}(uv) = \text{cnt}(u) \cup \text{cnt}(v).$$

We introduce the notion of "elementary medial transformation" of a word in A^+ analogously as for the left commutative groupoids ([1]).

Definition 1. Let $w \in A^+$, $w = a_1a_2\ldots a_n$ where $a_1,a_2,\ldots,a_n \in A$. A ***segment*** of w is called any expression of the form $a_ia_{i+1}\ldots a_{i+j}$, where $i+j \le n$, i runs in $\{1,2,\ldots, n\}$ and j runs in $\{0,1,\ldots, n-1\}$. ***The set of segments*** of w is denoted by seg(w). Thus:

$$\text{seg}(w) = \{a_ia_{i+1}\ldots a_{i+j} \mid i+j \le n,\ i\in\{1,2,\ldots, n\},\ j\in\{0,1,\ldots, n-1\}\}.$$

Proposition 1. If $w = a_1a_2\ldots a_n$, then $n \le |\text{seg}(w)| \le (n(n-1))/2$. □

Definition 2. Let $w \in A^+$, $|w| \ge 4$ and $txyz$ be a segment of w. If one occurrence of $txyz$ in w is replaced by $tyxz$, an element $u \in A^+$ is obtained that is different from w only in the replaced segment $txyz$ by $tyxz$. In that case we say that an ***elementary medial transformation*** is performed on w and write $w \rightharpoonup u$. In the cases when $w = a$ or $w = ab$ or $w = abc$, where a, b, $c \in A$, we put by definition $w \rightharpoonup w$. Thus, w = $\alpha txyz\beta \rightharpoonup \alpha tyxz\beta = u$, where $\alpha, \beta \in A^+$ or, α or β is the empty symbol.

We can "go back" from u to w performing the elementary medial transformation $u = \alpha tyxz\beta \rightharpoondown \alpha txyz\beta = w$ in the same place. We call this transformation an ***inverse return*** from u to w. Thus, $w \rightharpoonup u \rightharpoondown w$.

Note that an elementary medial transformation of an arbitrary word w does not lead to a transformation on A^+, because the "image" of w is not uniquely determined. For instance, the word $w = abcde$, where a, b, c, d, $e \in A$, with one elementary medial transformation can be transformed in $acbde$, but also in $w = abdce$.

Definition 3. Define a relation ~ on A^+ by:

$$w \sim u \Leftrightarrow (\exists w_0, w_1,\ldots,w_k \in A^+)\ w = w_0 \rightharpoonup w_1 \rightharpoonup \ldots \rightharpoonup w_{k-1} \rightharpoonup w_k = u.$$

Proposition 2. $w \sim u$, where $w = a_1a_2\ldots a_n$, $u = b_1b_2\ldots b_m$ if and only if $n = m$, $a_1 = b_1$, $a_n = b_n$ and $b_2\ldots b_{n-1}$ is a permutation of $a_2\ldots a_{n-1}$.

Proof. If $w \sim u$, then $|w|=|u|$, cnt(w) = cnt(u), left and right ends of w and u are equal and the appearance of any $a \in A$ in w is equal to the appearance of a in u. Therefore, $n = m$, $a_1 = b_1$, $a_n = b_n$ and $b_2\ldots b_{m-1}$ is a permutation of $a_2\ldots a_{n-1}$. Conversely, for $n = 1$, $n = 2$ and $n = 3$, $w \sim u$ is trivially fulfilled. Suppose that $w \sim u$, for $n \le k$ and $k \ge 4$. If $n = k+1$, then

$m = k+1$, $a_1 = b_1$, $a_{k+1} = b_{k+1}$ and so $u = a_1b_2\dots b_ka_{k+1}$. By the inductive supposition we obtain that $b_2\dots b_k \sim a_2\dots a_k$, and thus $a_1a_2\dots a_{k+1} \rightharpoonup a_1b_2\dots b_ka_{k+1}$, i.e. $w \sim u$. □

Directly from the definition of ~ it follows that ~ is an equivalence relation. If $w \sim u$ and $v \in A^+$, then $wv \sim uv$ and $vw \sim vu$, i.e. ~ is a congruence relation in A^+.

Define an operation $\cdot$ on $A^+/\sim = \{w^\sim \mid w \in A^+\}$ by:

$$w^\sim, u^\sim \in A^+/\sim \;\; \Rightarrow \;\; w^\sim \cdot u^\sim = (wu)^\sim.$$

It is clear that $(A^+/\sim,\cdot)$ is a semigroup. Further on we write $A^+/\sim$ instead of $(A^+/\sim,\cdot)$. Since $w_1^\sim \cdot w_2^\sim \cdot w_3^\sim \cdot w_4^\sim = (w_1w_2)^\sim \cdot (w_3w_4)^\sim =$ $(w_1w_2w_3w_4)^\sim = (w_1w_3w_2w_4)^\sim = (w_1w_3)^\sim \cdot (w_2w_4)^\sim = w_1^\sim \cdot w_3^\sim \cdot w_2^\sim \cdot w_4^\sim$, for any $w_1^\sim$, $w_2^\sim$, $w_3^\sim$, $w_4^\sim \in A^+/\sim$, the semigroup $(A^+/\sim,\cdot)$ is medial. Note that $a^\sim = \{a\}$, for any $a \in A$.

If $w^\sim \in A^+/\sim$, then $w^\sim = (a_1a_2\dots a_n)^\sim = a_1^\sim \cdot a_2^\sim \cdot \dots \cdot a_n^\sim$. Thus, $w^\sim$ is presented as a product of elements of $A/\sim$ and therefore $A/\sim$ is a generating set for $A^+/\sim$. Moreover, $A/\sim$ is a set of prime elements in $A^+/\sim$. Namely, if $a^\sim \in A/\sim$, then $a^\sim \neq w^\sim u^\sim$, for any $w^\sim, u^\sim \in A^+/\sim$, since a is not a product of elements of A^+. There are no other prime elements in $A^+/\sim$ except those of $A/\sim$. Namely, if $w^\sim \in (A^+/\sim) \setminus (A/\sim)$, then $w^\sim \notin (A/\sim)$, i.e. $w \notin A$. Thus, there are $u, v \in A^+$ such that $w = uv$. In that case, $w^\sim = (uv)^\sim = u^\sim \cdot v^\sim$, i.e. $w^\sim$ is not prime in $A^+/\sim$.

$A^+/\sim$ has the universal mapping property for *Med* over $A/\sim$, i.e. if $\boldsymbol{S} \in Med$ and $\lambda : A/\sim \to S$, then there is a homomorphism ψ from $A^+/\sim$ into $\boldsymbol{S}$ that is an extension of λ. Define the mapping $\psi : A^+/\sim \to S$ by $\psi(w^\sim) = \varphi(w)$, where φ is the homomorphism from A^+ in $\boldsymbol{S}$ that is an extension of λ.

The mapping ψ is well defined, that follows by the implication $w \sim u \Rightarrow \varphi(w) = \varphi(u)$. Namely, let $w \sim u$, i.e. $w_0, w_1, \dots, w_k \in A^+$ are such that $w = w_0 \rightharpoonup w_1 \rightharpoonup \dots \rightharpoonup w_{k-1} \rightharpoonup w_k = u$ and let $w_i = \alpha txyz \beta$, where $\alpha, \beta \in A^+$ or, α or β is the empty symbol. Since φ is a homomorphism it follows that:

$$\varphi(w_i) = \varphi(\alpha txyz\beta) = \varphi(\alpha)\varphi(t)\varphi(x)\varphi(y)\varphi(z)\varphi(\beta) = [S \in Med] =$$
$$= \varphi(\alpha)\varphi(t)\varphi(y)\varphi(x)\varphi(z)\varphi(\beta) = \varphi(\alpha tyxz\beta) = \varphi(w_{i+1}),$$

for $i \in \{0,...,k-1\}$. Hence, $\varphi(w) = \varphi(w_0) = \varphi(w_1) = ... = \varphi(w_k) = \varphi(u)$. The mapping ψ is a homomorphism from $A^+/\sim$ into ***S***, since

$$\psi(w^{\sim} \cdot u^{\sim}) = \psi((wu)^{\sim}) = \varphi(wu) = \varphi(w)\varphi(u) = \psi(w^{\sim})\psi(u^{\sim}).$$

By the above discussion we obtain that the following theorem holds.

Theorem 1. The quotient semigroup $(A^+/\sim,\cdot)$ is a free medial semigroup over $A^+/\sim$. □

Below we give a canonical description of a free medial semigroup.

Definition 4. Let *V* be a variety of semigroups. A semigroup $\boldsymbol{R} = (R,*)$ is said to be ***canonical*** in *V*, i.e. *V-**canonical***, over *A* if the following conditions are satisfied:

(c_0) $A \subseteq R \subseteq A^+$ and $w \in R \Rightarrow \mathrm{seg}(w) \subseteq R$;

(c_1) $wu \in R \Rightarrow w * u = wu$;

(c_2) ***R*** is a free semigroup in *V* over A.

Definition 5. Let the alphabet A be linearly ordered by ≤. A word $a_1a_2\ldots a_n \in A^+$ is said to be ***monotonic***, i.e. it has ***a natural order*** if and only if $a_1 \le a_2 \le \ldots \le a_n$.

As a candidate for the carrier of a free medial semigroup we choose the subset *R* of A^+ defined in the following way:

(i) $A \cup A^2 \cup A^3 \subseteq R$;

(ii) $(\forall w = a_1...a_n \in A^+, n \ge 4)(w \in R \Rightarrow a_2...a_{n-1} \text{ is monotonic})$.

By the definition of *R* and by induction on length of a word, one can obtain that the condition (c_0) is fulfilled. Note that among all the words of letters $a_1, a_2, ..., a_n$, only one is monotonic, i.e. only one has the natural order. We denote by $(a_1a_2...a_n)'$ the permutation of $a_1, a_2, ..., a_n$ that gives their natural order.

Define an operation $*$ on R by:

$$w = a_1a_2...a_n,\ u = b_1b_2...b_m \in R \ \Rightarrow\ w * u = a_1(a_2...a_nb_1...b_{m-1})'b_m.$$

Theorem 2. $\boldsymbol{R} = (R,*)$ is a Med-canonical semigroup over *A*. □

Proof. The permutation $(a_2...a_nb_1...b_{m-1})'$ gives the natural order of the elements $a_2,...,a_n,b_1,...,b_{m-1}$ and therefore $w * u$ is a uniquely determined word in ***R***, i.e. $*$ is a well-defined operation. Since $w * u = wu$, the condition (c_1) is fulfilled.

Let $w_1 = a_1a_2...a_n$, $w_2 = b_1b_2...b_m$, $w_3 = c_1c_2...c_k$, $w_4 = d_1d_2...d_p$.

Verifying the condition (c_2), we obtain:

$$(w_1 * w_2) * w_3 = a_1(a_2...a_nb_1b_2...b_{m-1})'b_m * c_1c_2...c_k =$$
$$= a_1(a_2...a_nb_1...b_{m-1}b_mc_1...c_{k-1})'c_k =$$
$$a_1...a_n * b_1(b_2...b_{m-1}b_mc_1...c_{k-1})'c_k =$$

$= w_1 * (w_2 * w_3)$. Therefore, $\boldsymbol{R} = (R,*)$ is a semigroup.

$\boldsymbol{R} = (R,*)$ is medial. Namely,

$$w_1 * w_2 * w_3 * w_4 = a_1a_2...a_nb_1b_2...b_mc_1c_2...c_kd_1d_2...d_{p-1}d_p =$$
$$= a_1(a_2...a_nb_1b_2...b_mc_1c_2...c_kd_1d_2...d_{p-1})'d_p =$$
$$= a_1(a_2...a_nc_1c_2...c_kb_1b_2...b_md_1d_2...d_{p-1})'d_p =$$
$$= a_1a_2...a_nc_1c_2...c_kb_1b_2...b_md_1d_2...d_{p-1}d_p = w_1 * w_3 * w_2 * w_4$$

If $w = a_1a_2...a_n \in R$, then $w = a_1 * a_2 * ... * a_n$ and thus, *A* is a generating set for ***R***. Moreover, *A* is the set of prime elements in ***R***. Namely, if $a \in A$, then $a \neq w * u$, for any $w,u \in R$, i.e. every element of *A* is prime in ***R***. If $w = a_1a_2...a_n \in R \setminus A$, then $n \geq 2$ and $a_1a_2...a_n = a_1 * a_2 * ... * a_n$, which means that *w* is a product in ***R***. There are no other prime elements in ***R*** except those in *A*.

R has the universal mapping property for Med over *A*, i.e. if $\boldsymbol{S} \in Med$ and $\lambda : A \to S$ is a mapping, then there is a homomorphism ψ from ***R*** into ***S*** such that ψ is an extension of λ. Define the mapping ψ from ***R*** into ***S*** by $\psi(w) = \varphi(w)$, for any $w \in R$, where φ is the homomorphism from A^+ in ***S*** that is an extension of λ. It is sufficient to show that $\psi(w * u) = \varphi(w)\varphi(u)$, for any $w = a_1a_2...a_n$, $u = b_1b_2...b_m$ in ***R.*** Namely,

$$\psi(w * u) = \varphi(a_1(a_2...a_nb_1b_2...b_{m-1})'b_m) = \varphi(a_1c_2...c_nc_{n+1}...c_{n+m-1}b_m) =$$
$$= \varphi(a_1)\varphi(c_2)...\varphi(c_n)\varphi(c_{n+1})...\varphi(c_{n+m-1})\varphi(b_m) =$$
$$= [S \in Med] = \varphi(a_1)\varphi(a_2)...\varphi(a_n)\varphi(b_1)...\varphi(b_{m-1})\varphi(b_m) =$$
$$= \varphi(a_1...a_n)\varphi(b_1...b_m) = \varphi(w)\varphi(u).$$

The conditions (c_0), (c_1) and (c_2) are satisfied. Hence, $\boldsymbol{R} = (R,*)$ is a Med-canonical semigroup over *A*. □

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