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*Research article*

# Abelian and Tauberian results for the fractional Fourier cosine (sine) transform

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**Abstract:** In this paper, we presented Tauberian type results that intricately link the quasi-asymptotic behavior of both even and odd distributions to the corresponding asymptotic properties of their fractional Fourier cosine and sine transforms. We also obtained a structural theorem of Abelian type for the quasi-asymptotic boundedness of even (resp. odd) distributions with respect to their fractional Fourier cosine transform (FrFCT) (resp. fractional Fourier sine transform (FrFST)). In both cases, we quantified the scaling asymptotic properties of distributions by asymptotic comparisons with Karamata regularly varying functions.

**Keywords:** fractional Fourier cosine (sine) transform; distributions; Abelian and Tauberian theorems
**Mathematics Subject Classification:** 40E05, 43A50, 46F10, 46F12

## 1. Introduction

The fractional Fourier transform (FrFT), a broader version of the traditional Fourier transform (FT), was introduced seven decades ago by Namias [13]. However, it has only recently gained traction in fields such as signal processing, optics, and quantum mechanics [3,11,26]. Notably, it can be employed with real-world data such as one-dimensional signals (e.g., electrocardiogram or ECG data) and two-dimensional signals like geographical information (e.g., satellite images) [10]. Many scientists defined

FrFT differently and enriched its theory properties [14]. Specially, in [7], FrFT of real order has been introduced using the Mittag-Leffler function. This transform plays the same role for the fractional derivatives as FT plays for the ordinary derivatives and is reduced into the FT, particularly for $\alpha = 1$ in the usual sense. FrFT is parameterized by $\alpha$ and effectively rotates a signal by an angle $\alpha$ within the time-frequency plane. Offering versatility, the FrFT facilitates the transformation of functions into various intermediate domains lying between time and frequency.

Alongside the FT, cosine (CT) and sine (ST) transforms play crucial roles in signal processing by expanding functions over cosine and sine basis functions, despite some differences compared to the FT. The idea of fractionalization of the CT and ST was proposed in [9]. There, authors selected the real and imaginary components of the FrFT kernel to function as the kernels for the fractional Fourier cosine transform (FrFCT) and the fractional Fourier sine transform (FrFST), respectively. However, they acknowledged that their fractional transforms lack index additivity and do not qualify as genuine fractional versions of CT and ST. In [2], FrFCT and FrFST that are additive on the index and preserve the similar relationships with the fractional FT were introduced. Their fractional FT corresponds to a rotation of the Wigner distribution and the ambiguity function. Subsequently, in [16] the discrete version of the FrFCT and FrFST based on the eigen decomposition of discrete CT and discrete ST kernels are defined. Since then, FrFST and FrFSTs have significantly evolved, expanded to spaces of generalized functions [18], and became powerful tools in mathematical analysis, physics, signal processing, etc. [1, 20, 21].

Distribution theory is a power tool in applied mathematics and the extension of integral transforms to generalized function spaces is an important subject, especially recently, when new transforms have been discovered and their connection with the old ones has been established. When studying the theory of distributions, one quickly learns that distributions do not have point values, which naturally imposes the idea of incorporating the asymptotic analysis to the field of generalized functions. The concept of quasi-asymptotics, as introduced in [25], serves to extend classical asymptotic methods within the framework of Schwartz distributions, finding applications across various fields, particularly in mathematical physics. There are many results where the asymptotic behavior of distributions is analyzed true to the behavior of various integral transforms in a form of the Abel and Tauberian type theorems, see [8, 19, 22, 23] and references therein.

The primary objective of this paper is to employ the FrFCT (FrFST) for a thorough investigation into the quasi-asymptotic properties of even (odd) distributions. Our study is structured around several theorems of the Abelian and Tauberian type, utilizing the asymptotic behavior of the FrFCT (FrFST) to investigate the quasi-asymptotic behavior of even (odd) distributions. The first section introduces the spaces of even (odd) tempered distributions. Subsequently, the FrFCT and FrFST are introduced through the FrFT and the Fourier cosine (sine) transform, establishing connections between all three transforms. The main result, presented in Section 2 with Theorem 2.3, asserts that if an even (odd) tempered distribution exhibits quasi-asymptotics at zero, then its FrFCT (FrFST) quasi-asymptotically oscillates at infinity. With an additional boundedness assumption, we demonstrate the converse. The second result, presented in Theorem 2.10, establishes the boundedness of the FrFCT (FrFST) of a distribution, assuming the distribution is quasi-asymptotically bounded.

*1.1. Notation and spaces*

We employ the standard notation $\mathcal{S}(\mathbb{R})$ to denote the space of rapidly decreasing smooth functions $f$ satisfying the condition:

$$\rho(f)_k = \sup_{t\in\mathbb{R},p\leqslant k}(1+|t|^2)^{k/2}|f^{(p)}(t)| < \infty, \quad k \in \mathbb{N}_0 = \mathbb{N}\cup 0. \tag{1.1}$$

The dual space of $\mathcal{S}(\mathbb{R})$ is the space of tempered distributions, denoted by $\mathcal{S}'(\mathbb{R})$. The subsets $\mathcal{S}_e(\mathbb{R})$ and $\mathcal{S}_o(\mathbb{R})$ consist of all even and odd functions, respectively, within $\mathcal{S}(\mathbb{R})$.

An example of an element in $\mathcal{S}_e(\mathbb{R})$ is given by $e^{-x^2}$, as $e^{-x^2} \in \mathcal{S}(\mathbb{R})$ and is even. Conversely, $xe^{x^2}$ belongs to $\mathcal{S}_o(\mathbb{R})$ since, according to [12], if $\varphi$ is an even differentiable function, then $C\varphi'$ is odd for any constant $C$.

An even (odd) tempered distribution is defined as a continuous linear functional on the vector space $\mathcal{S}_e(\mathbb{R})$ ($\mathcal{S}_o(\mathbb{R})$). The spaces of such distributions are denoted as $\mathcal{S}'_e(\mathbb{R})$ and $\mathcal{S}'_o(\mathbb{R})$, respectively. It is noteworthy that $\mathcal{S}'_e(\mathbb{R})$ and $\mathcal{S}'_o(\mathbb{R})$ constitute broader classes than $\mathcal{S}'(\mathbb{R})$, and specifically, $\mathcal{S}'_e(\mathbb{R}) \supset \mathcal{S}'(\mathbb{R})$ and $\mathcal{S}'_o(\mathbb{R}) \supset \mathcal{S}'(\mathbb{R})$. Moreover,

$$\mathcal{S}'_e(\mathbb{R}) \cap \mathcal{S}'_o(\mathbb{R}) = \mathcal{S}'(\mathbb{R}).$$

The FT of a function $f \in \mathcal{S}(\mathbb{R})$ is defined:

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-ix\xi}dx. \tag{1.2}$$

The map $f \to \hat{f}$ is a continuous bijection from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$, and can be extended by duality to $\mathcal{S}'(\mathbb{R})$.

For an even function $f$, the Fourier cosine transform (FCT) is given by

$$\hat{f}_c(\xi) = \mathcal{F}^c(f)(\xi) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos(t\xi)dt, \ \ \xi \in \mathbb{R},$$

and for an odd function $f$, the Fourier sine transform (FST) is given by

$$\hat{f}_s(\xi) = \mathcal{F}^s(f)(\xi) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin(t\xi)dt, \ \ \xi \in \mathbb{R}.$$

If $\phi \in \mathcal{S}_e(\mathbb{R})(\mathcal{S}_o(\mathbb{R}))$, then $\mathcal{F}^c(\phi) \in \mathcal{S}_e(\mathbb{R})(\mathcal{F}^s(\phi) \in \mathcal{S}_o(\mathbb{R}))$. The mapping $\mathcal{F}^c(\mathcal{F}^s)$ is a continuous isomorphism from $\mathcal{S}_e(\mathbb{R})(\mathcal{S}_o(\mathbb{R}))$ to $\mathcal{S}_e(\mathbb{R})(\mathcal{S}_o(\mathbb{R}))$ [5].

The development of the FrFCT and FrFST of distributions follows the conventional approach employed in FTs. Consequently, it is necessary to highlight certain fundamental properties of the FrFCT and FrFST for functions (refer to [12, Chapter 3]).

The FrFCT (FrFST) of an even (odd) tempered distribution $f$ can be defined by

$$\langle \mathcal{F}^c f, \varphi\rangle = \langle f, \mathcal{F}^c\varphi\rangle, \quad (\langle \mathcal{F}^s f, \varphi\rangle = \langle f, \mathcal{F}^s\varphi\rangle)$$

for arbitrary $\varphi \in \mathcal{S}_e(\mathbb{R})(\mathcal{S}_o(\mathbb{R}))$. Thus, $\mathcal{F}^c(\mathcal{F}^s)$ is a continuous mapping from $\mathcal{S}'_e(\mathbb{R})(\mathcal{S}'_o(\mathbb{R}))$ to $\mathcal{S}'_e(\mathbb{R})(\mathcal{S}'_o(\mathbb{R}))$.

### 1.2. Fractional Fourier transform

Let $f \in L^1(\mathbb{R})$. Recall from [13] that the FrFT of order $\alpha$ is defined by

$$\mathcal{F}_\alpha(f(x))(\xi) = \begin{cases} \int_{\mathbb{R}} f(x)K_\alpha(x,\xi)dx, & \xi \in \mathbb{R}, \quad \alpha \neq k\pi, k \in \mathbb{N}, \\ f(\xi), & \alpha = 2k\pi, \\ f(-\xi), & \alpha = (2k+1)\pi, \end{cases} \tag{1.3}$$

where

$$K_\alpha(x,\xi) = C_\alpha e^{i(\frac{x^2+\xi^2}{2}a - x\xi b)} \tag{1.4}$$

is the kernel of FrFT, $a = \cot\alpha$, $b = \csc\alpha$, and $C_\alpha = \sqrt{\frac{1-ia}{2\pi}}$. The kernel $K_\alpha(x,\xi)$ is a continuously differentiable function in both variables, $x$ and $\xi$. The function $\mathcal{F}_\alpha f$ exhibits $2\pi$ periodicity with respect to $\alpha$, and, thus, we will consistently consider $\alpha$ within the interval $[0, 2\pi)$. Notice that when $n \in \mathbb{Z}$, $\mathcal{F}_{n\pi/2}f = \mathcal{F}^n f$, where $\mathcal{F}^n$ is the $n$-th power of the FT (1.2), i.e., $\mathcal{F}_\alpha$ is the $s$th power of the FT for $s = 2\alpha/\pi$, for $\alpha$ within the interval $[0, 2\pi)$, [13]. So, the FT is of order 1, while the identity operator is of order 0. Negative orders correspond to inverse transforms. For instance, applying the FrFT of order 1/2 twice yields the FT.

In the research conducted by Pathak [15, Thrm. 3.1], it has been established that the FrFT constitutes a continuous mapping from the Schwartz space $\mathcal{S}(\mathbb{R})$ to itself. Furthermore, this mapping can be extended to the space of tempered distributions. Specifically, according to the definition provided in [15, Def. 3.1], the generalized FrFT, denoted as $\mathcal{F}_\alpha f$ for $f \in \mathcal{S}'(\mathbb{R})$, is expressed as follows:

$$\langle \mathcal{F}_\alpha f, \varphi \rangle = \langle f, \mathcal{F}_\alpha \varphi \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}). \tag{1.5}$$

For $f \in \mathcal{S}(\mathbb{R})$, the inverse FrFT $\mathcal{F}_{-\alpha}$ is given with [3]

$$f(x) = \int_{\mathbb{R}} \mathcal{F}_\alpha f(\xi) K_{-\alpha}(x,\xi) d\xi, \qquad x \in \mathbb{R}. \tag{1.6}$$

From the linearity of the FrFT and the reversion property $\mathcal{F}_\alpha(f(-x))(\xi) = \mathcal{F}_\alpha(f(x))(-\xi)$, we have

$$\mathcal{F}_\alpha(f(x) \pm f(-x))(\xi) = \mathcal{F}_\alpha(f(x))(\xi) \pm \mathcal{F}_\alpha(f(x))(-\xi),$$

and we conclude that the FrFT of an even function is even, while the FrFT of an odd function is odd.

### 1.3. The fractional Fourier cosine (sine) transform

In [6], the FrFCT is denoted as $\mathcal{F}_\alpha^c(f(x))(\xi) = \int_{-\infty}^{\infty} f(x)K_\alpha^c(x,\xi)dx$, and the FrFST is represented as $\mathcal{F}_\alpha^s(f(x))(\xi) = \int_{-\infty}^{\infty} f(x)K_\alpha^c(x,\xi)dx$. It is important to observe that if the function $f$ is odd, then $\mathcal{F}_\alpha^c(f(x))(\xi) = 0$. Similarly, if $f$ is an even function, then the corresponding FrFCT simplifies to the one-sided FrFCT, given by:

$$\mathcal{F}_\alpha^c(f(x))(\xi) = \int_0^{\infty} f(x)K_\alpha^c(x,\xi)dx.$$

A similar consideration can also be repeated for the FrFST.

Here, we follow the notion from [3,18], and the definitions from there. This means that if we restrict ourself to one-sided functions ($f(x) = 0$ for $x < 0$), we can define the FrFCT of a function $f \in L^1(\mathbb{R})$ as

$$\mathcal{F}_\alpha^c(f(x))(\xi) = \mathcal{F}_\alpha(f(x) + f(-x))(\xi) = \int_0^\infty f(x)K_\alpha^c(x,\xi)dx, \tag{1.7}$$

where

$$K_\alpha^c(x,\xi) = \begin{cases} C_\alpha e^{i\frac{x^2+\xi^2}{2}a}\cos(x\xi b), & \alpha \neq k\pi, k \in \mathbb{Z} = \mathbb{N}_0 \cup (-\mathbb{N}), \\ \sqrt{\frac{2}{\pi}}\cos(x\xi), & \alpha = \pi/2, \\ \delta(x-\xi), & \alpha = k\pi, \end{cases} \tag{1.8}$$

and $a = \cot\alpha,\ b = \csc\alpha$, and $C_\alpha = \sqrt{\frac{2(1-ia)}{\pi}}$.

The inverse FrFCT is given by

$$f(x) = \int_0^\infty \mathcal{F}_\alpha^c f(\xi)K_{-\alpha}^c(x,\xi)d\xi, \qquad x \in \mathbb{R}. \tag{1.9}$$

Similarly, the FrFST of a function $f \in L^1(\mathbb{R})$ is defined as

$$\mathcal{F}_\alpha^s(f(x))(\xi) = \mathcal{F}_\alpha(f(x) - f(-x))(\xi) = \int_0^\infty f(x)K_\alpha^s(x,\xi)dx, \tag{1.10}$$

where

$$K_\alpha^s(x,\xi) = \begin{cases} C_\alpha e^{i(\alpha-\pi/2)\frac{x^2+\xi^2}{2}a}\sin(x\xi b), & \alpha \neq k\pi, k \in \mathbb{Z} = \mathbb{N}_0 \cup (-\mathbb{N}), \\ \sqrt{\frac{2}{\pi}}\sin(x\xi), & \alpha = \pi/2, \\ \delta(x-\xi), & \alpha = k\pi, \end{cases} \tag{1.11}$$

with the same constants as above. The corresponding inverse FrFST is given by

$$f(x) = \int_0^\infty \mathcal{F}_\alpha^s f(\xi)K_{-\alpha}^s(x,\xi)d\xi, \qquad x \in \mathbb{R}. \tag{1.12}$$

The kernel $K_\alpha^c(x,\xi)(K_\alpha^s(x,\xi))$ is a continuously differentiable function in both variables, $x$ and $\xi$. For $\alpha = \frac{(2k-1)\pi}{2}$, the FrFCT and FrFST are reduced to the FCT and FST, respectively.

To establish the connection between the FrFT of a causal, one-sided function $f(x)$, where $f(x) = 0$ for $x < 0$, and the FrFCT and FrFST of this function in an alternative manner, we can formulate the following expression.

$$2\mathcal{F}_\alpha f(\pm\xi) = \mathcal{F}_\alpha^c f(\xi) \pm e^{i\alpha}\mathcal{F}_\alpha^s f(\xi).$$

The expression $\mathcal{F}_\alpha^c f(\xi)$ can be associated with the even component of $\mathcal{F}_\alpha f(\xi)$, while $\mathcal{F}_\alpha^s f(\xi)$ is linked to its odd component. In a broader context, it can be concluded that for determining the FrFCT of a causal, one-sided function $f(x)$, one can equivalently determine the FrFT of the symmetrically extended two-sided function $f(x) + f(-x)$. Similarly, to ascertain the FrFST of such a function, one can alternatively determine the FrFT of the anti-symmetrically extended two-sided function $e^{j\alpha}(f(x) - f(-x))$. Both cases involve restricting the analysis to $\xi \geqslant 0$. Moreover, $\mathcal{F}_\alpha^c f(\xi)$ and $\mathcal{F}_\alpha f(\xi)$ are periodic with period $\pi$ [2].

In [6], authors studied the FrFCT (resp., FrFST) for the space $\mathcal{S}_e(\mathbb{R})$ (resp., $\mathcal{S}_o(\mathbb{R})$). They demonstrated a Parseval-type relationship, an inversion formula, and the continuity properties of the FrFCT and FrFST. They, and also Prasad and Sihgh in [18, Thrm 3.1 and Thrm 3.2], have shown that the FrFCT is the continuous linear mapping of $\mathcal{S}_e(\mathbb{R})$ onto itself, and that the FrFST is the continuous linear mapping of $\mathcal{S}_o(\mathbb{R})$ onto itself. This allows them to define the generalized FrFCT (FrFST) of distribution $f$ from $\mathcal{S}'_e(\mathbb{R})$ ($\mathcal{S}'_o(\mathbb{R})$) by

$$\langle \mathcal{F}^c_\alpha f, \varphi\rangle = \langle f, \mathcal{F}^c_\alpha \varphi\rangle, \quad (\langle \mathcal{F}^s_\alpha f, \varphi\rangle = \langle f, \mathcal{F}^s_\alpha \varphi\rangle) \tag{1.13}$$

for all $\varphi \in \mathcal{S}_e(\mathbb{R})(\mathcal{S}_o(\mathbb{R})))$. Moreover, the FrFCT (FrFST) of a distribution $f \in \mathcal{S}'_e(\mathbb{R})(\mathcal{S}'_o(\mathbb{R}))$ is a continuous linear map of $\mathcal{S}'_e(\mathbb{R})(\mathcal{S}'_o(\mathbb{R}))$ onto itself.

We need the relation between the FrFCT and the FCT:

$$\begin{aligned}\mathcal{F}^c_\alpha f(\xi) = \int_0^\infty f(x) K^c_\alpha(x,\xi)dx &= C_\alpha e^{i\xi^2 a/2}\int_0^\infty e^{ix^2 a/2}\cos(x\xi b) f(x)dx \\ &= \sqrt{1-ia}e^{i\xi^2 a/2}\mathcal{F}^c(e^{ix^2 a/2} f(x))(\xi b).\end{aligned} \tag{1.14}$$

Similarly, it holds for the FrFST and the FST.

## 2. Asymptotic behavior of distributions

### 2.1. Quasi-asymptotic behavior for $\mathcal{F}^c_\alpha f$ ($\mathcal{F}^s_\alpha f$), $f \in \mathcal{S}'_e$ ($\mathcal{S}'_o$)

We will analyze the properties of a distribution by comparing it to regularly varying functions, specifically focusing on the quasi-asymptotic behavior outlined in [17, 22, 23]. A real-valued function, measurable, defined, and positive within an interval $(0, D]$ (or $[D, \infty)$), where $D > 0$, is termed “a slowly varying function” at the origin (or at infinity) if it satisfies:

$$\lim_{\varepsilon\to 0^+}\frac{L(d\varepsilon)}{L(\varepsilon)} = 1 \quad (\text{ resp.} \lim_{h\to\infty}\frac{L(dh)}{L(h)} = 1) \quad \text{for each } d > 0. \tag{2.1}$$

If $L$ is a slowly varying function at zero, then $\tilde{L}(\cdot) = L(1/\cdot)$ is also a slowly varying function in a neighborhood of $\infty$, and vice versa.

Considering $L$ as a function exhibiting slow variation at the origin, it is relevant to recall the definition from [4] that a distribution $f \in \mathcal{S}'(\mathbb{R})$ manifests quasi-asymptotic behavior, or quasi-asymptotics, of degree $m \in \mathbb{R}$ at the point $x_0 \in \mathbb{R}$ concerning $L$. This characterization is established if there exists $u \in \mathcal{S}'(\mathbb{R})$ such that, for every $\varphi \in \mathcal{S}(\mathbb{R})$, the following condition is satisfied:

$$\lim_{\varepsilon\to 0^+}\langle \frac{f(x_0+\varepsilon x)}{\varepsilon^m L(\varepsilon)}, \varphi(x)\rangle = \langle u(x), \varphi(x)\rangle. \tag{2.2}$$

The quasi-asymptotic behavior is conveniently denoted as:

$$f(x_0+\varepsilon x) \sim \varepsilon^m L(\varepsilon) u(x) \quad \text{as} \quad \varepsilon \to 0^+ \quad \text{in} \quad \mathcal{S}'(\mathbb{R}),$$

and should be consistently interpreted within the framework of the weak topology $\mathcal{S}'(\mathbb{R})$, i.e., in the sense of (2.2).

The form of $u$ is not arbitrary; it must exhibit homogeneity with a degree of homogeneity $m$, i.e., $u(dx) = d^m u(x)$, for all $d > 0$ [17, 25]. Additionally, if (2.2) holds for each $\varphi \in \mathcal{D}(\mathbb{R})$, it must also hold for each $\varphi \in \mathcal{S}(\mathbb{R})$ [22, Thrm. 6.1]. Consequently, the quasi-asymptotic behavior at finite points is considered a local property. The quasi-asymptotics of distributions at infinity concerning a slowly varying function $L$ at infinity is similarly defined. The notation $f(hx) \sim h^m L(h)u(x)$ as $h \to \infty$ in $\mathcal{S}'(\mathbb{R})$ is employed in this case.

Furthermore, we may explore quasi-asymptotics within alternative distribution spaces. The relationship $f(x_0 + \varepsilon x) \sim \varepsilon^m L(\varepsilon)u(x)$ as $\varepsilon \to 0^+$ in $\mathcal{B}'(\mathbb{R})$ implies that (2.2) holds for each $\varphi \in \mathcal{B}(\mathbb{R})$. Similarly, the quasi-asymptotics at infinity in $\mathcal{B}'(\mathbb{R})$ satisfy this condition, where $\mathcal{B}(\mathbb{R})$ denotes any space of test functions on $\mathbb{R}$, and $\mathcal{B}'(\mathbb{R})$ represents their dual.

The next lemma, as presented in [4, Lemma 3.1], establishes a connection between quasi-asymptotic behavior at the point and the corresponding oscillation at that same point.

**Lemma 2.1.** *If*

$$\langle f(\varepsilon x)/(\varepsilon^m L(\varepsilon)), \varphi(x)\rangle, \text{ converges as } \varepsilon \to 0^+, \ \forall \varphi \in \mathcal{S}, \tag{2.3}$$

then,

$$\langle e^{iC(\varepsilon x)^2/2} f(\varepsilon x)/(\varepsilon^m L(\varepsilon)), \varphi(x)\rangle, \ \text{ converges as } \varepsilon \to 0^+, \forall \varphi \in \mathcal{S}, \tag{2.4}$$

where $C \in \mathbb{R}$ and $\varepsilon \in (0, 1)$. On the contrary, if condition (2.4) is satisfied and there exists $\varepsilon_0 \in (0, \varepsilon)$ such that the family

$$\{f(\varepsilon x)/(\varepsilon^m L(\varepsilon)) : \varepsilon \in (0, \varepsilon_0)\} \text{ is bounded in } \mathcal{S}'(\mathbb{R}), \tag{2.5}$$

then (2.4) $\Rightarrow$ (2.3).

Moreover, (2.3) is equivalent to

$$\langle e^{-i\frac{h^2\xi^2}{2C}} * \widehat{f}(h\xi)/(h^{-m-2}\tilde{L}(h)), \gamma(\xi)\rangle, \text{ converges as } h \to \infty, \ \forall \gamma \in \mathcal{S}, \tag{2.6}$$

where $h > h_0 > 0$.

**Theorem 2.2.** *Let $f \in \mathcal{S}'_e(\mathbb{R})(\mathcal{S}'_o(\mathbb{R}))$. The statements below are equivalent:*

$$f(x/\varepsilon) \sim \varepsilon^m L(\varepsilon)u(x) \quad \text{as } \varepsilon \to \infty \quad \text{in} \quad \mathcal{S}'_e(\mathbb{R}^+)(\mathcal{S}'_o(\mathbb{R})), \tag{2.7}$$

and

$$\begin{aligned} \hat{f}_c(h\xi) &\sim h^{-m-1}\tilde{L}(h)\hat{u}_c(\xi) \quad \text{as } h \to \infty \text{ in } \mathcal{S}'_e(\mathbb{R}^+), \\ (\hat{f}_s(h\xi) &\sim h^{-m-1}\tilde{L}(h)\hat{u}_s(\xi) \quad \text{as } h \to \infty \text{ in } \mathcal{S}'_o(\mathbb{R}^+)), \end{aligned} \tag{2.8}$$

where $\tilde{L}(h)$ is slowly varying at $\infty$, $(\tilde{L}(\cdot) = L(\frac{1}{\cdot}))$.

*Proof.* Let $\varphi \in \mathcal{S}_e(\mathbb{R})$. The relation obtained by using the Parseval identity

$$\langle \frac{\hat{f}_c(h\xi)}{h^{-m-1}\tilde{L}(h)}, \varphi(\xi)\rangle = \langle \frac{\hat{f}_c(\xi)}{h^{-m}\tilde{L}(h)}, \varphi(\frac{\xi}{h})\rangle = \langle \frac{f(x)}{h^{-m}\tilde{L}(h)}, \hat{\varphi}_c(xh)\rangle = \langle \frac{f(\frac{x}{h})}{h^{-m}\tilde{L}(h)}, \hat{\varphi}_c(x)\rangle,$$

immediately implies the assertion.

The following theorem asserts that if $f \in \mathcal{S}'_e(\mathbb{R})(f \in \mathcal{S}'_o(\mathbb{R}))$ has quasi-asymptotics at zero, then $\mathcal{F}^c_\alpha f(\mathcal{F}^s_\alpha f)$ quasi-asymptotically oscillates as it approaches infinity. The reverse is true, with the additional assumption (2.5).

**Theorem 2.3.** *Let $f \in \mathcal{S}'_e(\mathbb{R})(\mathcal{S}'_o(\mathbb{R}))$, $L$ be of the form* (2.1) *at $0^+$, $u \in \mathcal{S}'_e(\mathbb{R})(\mathcal{S}'_o(\mathbb{R}))$ be a homogeneous function, and $m \in \mathbb{R}$. If*

$$f(\varepsilon x) \sim \varepsilon^m L(\varepsilon) u(x) \quad \text{as} \quad \varepsilon \to 0^+ \quad \text{in} \ \mathcal{S}'_e(\mathbb{R})(\mathcal{S}'_o(\mathbb{R})), \tag{2.9}$$

then

$$\begin{aligned} e^{-ia(h\xi)^2/2}\mathcal{F}^c_\alpha f(h\xi) &\sim \frac{\sqrt{1-ia}}{b^{m+1}} h^{-m-1}\tilde{L}(h)\widehat{u}_c(\xi) \quad \text{as} \quad h \to \infty, \\ (e^{-ia(h\xi)^2/2}\mathcal{F}^s_\alpha f(h\xi) &\sim \frac{\sqrt{1-ia}}{b^{m+1}} h^{-m-1}\tilde{L}(h)\widehat{u}_s(\xi) \quad \text{as} \quad h \to \infty), \end{aligned} \tag{2.10}$$

in $\mathcal{S}'_e(\mathbb{R})(\mathcal{S}'_o(\mathbb{R}))$. Conversely, if (2.5) holds, then (2.10) $\Rightarrow$ (2.9).

*Proof.* The notation with $1/\varepsilon$ will be maintained in the proof instead of using $h$. We will present the proof for the FrFCT, since the proof for the FrFST is analogous.

Let $\varphi \in \mathcal{S}_e(\mathbb{R})$. Using (1.13) and (1.14), we obtain:

$$\begin{aligned} &\langle e^{-ia(\frac{\xi}{\varepsilon})^2/2}\mathcal{F}^c_\alpha f(\frac{\xi}{\varepsilon}), \varphi(\xi)\rangle = \varepsilon\langle e^{-ia\xi^2/2}(\mathcal{F}^c_\alpha f)(\xi), \varphi(\varepsilon\xi)\rangle \\ &= \frac{\varepsilon C_\alpha\sqrt{2\pi}}{b}\langle \mathcal{F}^c(e^{iax^2/2}f(x))(\xi), \varphi(\frac{\varepsilon\xi}{b})\rangle = \sqrt{1-ia}\langle e^{iax^2/2}f(x), \widehat{\varphi}_c(\frac{xb}{\varepsilon})\rangle \\ &= \frac{\sqrt{1-ia}\varepsilon}{b}\langle e^{ia(\frac{\varepsilon t}{b})^2/2}f(\frac{\varepsilon t}{b}), \widehat{\varphi}_c(t)\rangle. \end{aligned}$$

The essential relation is the next one (see [4]):

$$\frac{1}{\varepsilon^{m+1}L(\varepsilon)}\langle e^{-ia(\frac{\xi}{\varepsilon})^2/2}\mathcal{F}^c_\alpha f(\frac{\xi}{\varepsilon}), \varphi(\xi)\rangle = \frac{L(\frac{\varepsilon}{b})}{L(\varepsilon)}\frac{\sqrt{1-ia}}{b^{m+1}(\frac{\varepsilon}{b})^m L(\frac{\varepsilon}{b})}\langle e^{ia(\frac{\varepsilon t}{b})^2/2}f(\frac{\varepsilon t}{b}), \widehat{\varphi}_c(t)\rangle. \tag{2.11}$$

From (2.1), we obtain:

$$\begin{aligned} &\lim_{\varepsilon\to 0^+}\langle \frac{e^{-ia(\frac{\xi}{\varepsilon})^2/2}\mathcal{F}^c_\alpha f(\frac{\xi}{\varepsilon})}{\varepsilon^{m+1}L(\varepsilon)}, \varphi(\xi)\rangle = \\ &= \frac{\sqrt{1-ia}}{b^{m+1}}\lim_{\varepsilon\to 0^+}\frac{1}{(\frac{\varepsilon}{b})^m L(\frac{\varepsilon}{b})}\langle e^{ia(\frac{\varepsilon t}{b})^2/2}f(\frac{\varepsilon t}{b}), \widehat{\varphi}_c(t)\rangle \\ &= \frac{\sqrt{1-ia}}{b^{m+1}}\left(\lim_{\varepsilon\to 0^+}\frac{1}{(\frac{\varepsilon}{b})^m L(\frac{\varepsilon}{b})}\langle f(\frac{\varepsilon t}{b}), \widehat{\varphi}_c(t)\rangle + \lim_{\varepsilon\to 0^+}\frac{1}{(\frac{\varepsilon}{b})^m L(\frac{\varepsilon}{b})}\langle f(\frac{\varepsilon t}{b}), (e^{ia(\frac{\varepsilon t}{b})^2/2}-1)\widehat{\varphi}_c(t)\rangle\right). \end{aligned}$$

The second limit is zero because

$$\lim_{\varepsilon\to 0^+}(e^{ia(\frac{\varepsilon t}{b})^2/2}-1)\widehat{\varphi}_c(t) = 0,$$

and $\{\frac{f(\frac{\varepsilon t}{b})}{(\frac{\varepsilon}{b})^m L(\frac{\varepsilon}{b})} : \varepsilon \in (0, \varepsilon_0)\}$ is bounded in $\mathcal{S}'_e(\mathbb{R})$.

Now, by (2.9) and the Banach Steinhaus theorem, we have

$$\lim_{\varepsilon\to 0^+}\langle \frac{e^{-ia(\frac{\xi}{\varepsilon})^2/2}\mathcal{F}^c_\alpha f(\frac{\xi}{\varepsilon})}{\varepsilon^{m+1}L(\varepsilon)}, \varphi(\xi)\rangle = \frac{\sqrt{1-ia}}{b^{m+1}} \lim_{\varepsilon\to 0^+}\langle \frac{f(\varepsilon t)}{\varepsilon^m L(\varepsilon)}, \widehat{\varphi}_c(t)\rangle$$
$$= \frac{\sqrt{1-ia}}{b^{m+1}}\langle u(x), \widehat{\varphi}_c(x)\rangle = \frac{\sqrt{1-ia}}{b^{m+1}}\langle \widehat{u}_c(\xi), \varphi(\xi)\rangle.$$

This proves (2.9) $\Rightarrow$ (2.10).

For the opposite implication, we assume that (2.10) holds and we use (2.11).

$$\lim_{\varepsilon\to 0^+}\langle \frac{1}{\varepsilon^{m+1}L(\varepsilon)} f(\varepsilon x), \varphi(x)\rangle$$
$$= \lim_{\varepsilon\to 0^+}\langle \frac{1}{\varepsilon^{m+1}L(\varepsilon)} e^{-ia(x\varepsilon)^2/2} f(\varepsilon x), \varphi(x)\rangle - \lim_{\varepsilon\to 0^+}\langle \frac{1}{\varepsilon^{m+1}L(\varepsilon)}(e^{-ia(x\varepsilon)^2/2} - 1) f(\varepsilon x), \varphi(x)\rangle.$$

Now, by the boundedness condition (2.5), we have that

$$\langle \frac{1}{\varepsilon^{m+1}L(\varepsilon)}(e^{-ia(x\varepsilon)^2/2} - 1) f(\varepsilon x), \varphi(x)\rangle \to 0, \text{ as } \varepsilon \to 0^+, \varphi \in \mathcal{S}_e.$$

For the last part, we note that for any $p > 0$:

$$\mathcal{F}^c(e^{\frac{ipx^2}{2}})(\xi) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{p}} e^{-i\frac{\xi^2}{2p}}, \xi \in \mathbb{R}^+.$$

Applying the FT to Eq (2.3), we find that there exists a suitable constant $\tilde{C} \in \mathbb{C}$ such that:

$$\langle \mathcal{F}^c(e^{iC(\varepsilon x)^2})(\xi) * \mathcal{F}^c(f(\varepsilon x))(\xi)/(\varepsilon^m L(\varepsilon)), (\hat{\varphi}_c)(\xi)\rangle$$
$$= \tilde{C}\frac{1}{\varepsilon^{m+2}L(\varepsilon)}\langle e^{-i\xi^2/(2\varepsilon^2 C)} * \mathcal{F}^c(f)(\xi/\varepsilon), (\hat{\varphi}_c)(\xi)\rangle.$$

If we put $h = 1/\varepsilon$ and $\gamma = \hat{\varphi}_c$, we obtain (2.6). The implication (2.6) $\Rightarrow$ (2.4) follows in the same way. The result is now obvious by (2.11).

*Remark* 2.4. Now, we take $\tan\beta = \varepsilon^2 \tan\alpha$, and $\varepsilon > 0$, with $C_{\alpha,\beta}$ defined as:

$$C_{\alpha,\beta}(\xi) = \sqrt{\frac{\cos\beta}{\cos\alpha}}\frac{e^{i\frac{\alpha}{2}}}{e^{i\frac{\beta}{2}}}\exp\left(i\frac{\xi^2}{2}\cot\alpha(1 - \frac{\cos^2\beta}{\cos^2\alpha})\right).$$

We have the following scalar property for the FrFCT (FrFST):

$$\mathcal{F}^c_\alpha(f(\varepsilon x))(\xi) = C_{\alpha,\beta}(\xi)\mathcal{F}^c_\beta(f(x))(\frac{s\xi}{\varepsilon}), \tag{2.12}$$

and

$$\mathcal{F}^s_\alpha(f(\varepsilon x))(\xi) = C_{\alpha,\beta}(\xi)e^{-i\beta}\mathcal{F}^s_\beta(f(x))(\frac{s\xi}{\varepsilon}), \tag{2.13}$$

where $s = \frac{\sin\beta}{\sin\alpha}$, [2], then, we can obtain the same result with very few modifications.

**Example 2.5.** It is known that every bounded function in $\mathbb{R}$ defines a tempered distribution, so the function $f(x) = H(x)(2 + \sin\frac{1}{x}) \in \mathcal{S}'(\mathbf{R})$, where $H(x)$ is Heviside's function. It is known that it does not have regular asymptotics at 0, but $f(\varepsilon x) \sim 2$ as $\varepsilon \to 0^+$ ($m = 0$, $L(\epsilon) = 1$ and $u(x) = 2$). Since $\widehat{u}_s(\xi) = \frac{2}{\xi}\sqrt{\frac{2}{\pi}}$, by Theorem 2.3 we have

$$e^{-ia(h\xi)^2/2}\mathcal{F}^s_\alpha f(h\xi) \sim \frac{1}{h\xi}\frac{2\sqrt{2(1-ia)}}{\sqrt{\pi}b} \quad \text{as} \quad h \to \infty$$

with respect to $\tilde{L}(\xi) \equiv 1$.

**Example 2.6.** For $f(x) = \sin x \in \mathcal{S}'_0(\mathbb{R})$, it is known that $f(\varepsilon x) \sim \varepsilon^{-1}\delta(x)$ as $\varepsilon \to 0^+$, for $m = 1, L(\varepsilon) \equiv 1, u(x) = \delta(x)$, then by $\widehat{u}_s(\xi) = \widehat{\delta}_s(\xi) = 0$ and Theorem 2.3, we have

$$e^{-ia(h\xi)^2/2}\mathcal{F}^s_\alpha f(h\xi) \sim 0 \quad \text{as} \quad h \to \infty$$

with respect to $\tilde{L}(\xi) \equiv 1$.

**Example 2.7.** Let $f(x) = 2\pi\delta(x-1)$, so it follows that $f(\varepsilon x) \sim \varepsilon\frac{2\pi}{\delta}(x-1)$ as $\varepsilon \to 0^+$, then by $\widehat{u}_s(\xi) = \mathcal{F}^s(2\pi\delta(\xi - 1)) = \sqrt{2\pi}e^{i\xi}$ and Theorem 2.3, we have

$$e^{-ia(h\xi)^2/2}\mathcal{F}^s_\alpha f(h\xi) \sim \sqrt{2\pi}\frac{\sqrt{(1-ia)}}{b^2}h^{-2}e^{i\xi} \quad \text{as} \quad h \to \infty$$

with respect to $\tilde{L}(\xi) \equiv 1$.

### 2.2. *Quasi-asymptotic boundednes of distributions*

We require an additional concept from quasi-asymptotic analysis, specifically the idea of *quasi-asymptotic boundedness* as in [24]. Recall from [24] that the distribution $f \in \mathcal{S}'(\mathbb{R})$ is quasi-asymptotically bounded at $x_0 \in \mathbb{R}$ of degree $m \in \mathbb{R}$, in relation to the slowly varying function $L$ at the origin if

$$f(x_0 + \varepsilon x) = O(\varepsilon^m L(\varepsilon)) \text{ as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R}).$$

The above relation should be interpreted in the sense of the weak topology of $\mathcal{S}'(\mathbb{R})$, namely,

$$\langle f(x_0 + \varepsilon x), \varphi(x)\rangle = O(\varepsilon^m L(\varepsilon)) \text{ as } \varepsilon \to 0^+, \tag{2.14}$$

for every $\varphi \in \mathcal{S}(\mathbb{R})$. This means there exists a constant $C > 0$ such that

$$|\langle f(x_0 + \varepsilon x), \varphi(x)\rangle| \leqslant C|\varepsilon^m L(\varepsilon)|,$$

for $x$ sufficiently close to $x_0$.

The notion of quasi-asymptotic boundedness, with respect to a degree $m \in \mathbb{R}$ in correlation with the slowly varying function at infinity $L$, is explicated in an analogous manner. Concurrently, our examination extends to the contemplation of quasi-asymptotic boundedness within a designated mathematical space: $\mathcal{S}'_e(\mathbb{R})$, $\varphi \in \mathcal{S}_e(\mathbb{R})$ ($\mathcal{S}'_o(\mathbb{R})$, $\varphi \in \mathcal{S}_o(\mathbb{R})$).

Please note that $L$ exhibits slow variation at the origin if, and only if, there exist measurable functions $u$ and $w$ defined on an interval $(0, A]$, where $u$ is bounded and possesses a finite limit at

0. Additionally, $\omega$ is continuous on $[0, A]$ with $\omega(0) = 0$, such that the following representation holds for $L(x)$ within the interval $(0, A]$:

$$L(x) = \exp\left(u(x) + \int_x^A \frac{\omega(\xi)}{\xi} d\xi\right), \qquad x \in (0, A].$$

In the context of our exploration for suitable modifications of $L$ concerning quasi-asymptotics, it is reasonable to assume that $L$ is defined across the entire interval $(0, \infty)$ and maintains nonnegativity or even positivity throughout. The analysis entails extending the functions $u$ and $\omega$ to $(0, \infty)$ through a chosen method.

For example, when addressing functions with slow variation at the origin, the condition $\xi^{-1}\omega(\xi) \in L^1([1, \infty))$ implies the existence of positive constants $\tilde{C}$ and $C$ such that the following inequalities hold for $x > 1$:

$$\tilde{C} < L(x) < C.$$

*Remark* 2.8. It is readily apparent that when $f$ belongs to the space $\mathcal{S}'_c(\mathbb{R})$ ($f$ belonging to $\mathcal{S}'_o(\mathbb{R})$) and exhibits quasi-asymptotic boundedness at zero concerning $\varepsilon^m L(\varepsilon)$, then $e^{iC(x)^2} f(x)$ also demonstrates quasi-asymptotic boundedness at zero, considering the same slowly varying function. Here, $x$ is a real number, and the condition $|e^{iC(x)^2}| = 1$ holds.

*Remark* 2.9. It is clear that for $\varphi \in \mathcal{S}_e(\mathbb{R})$, it follows that $x^p\varphi \in \mathcal{S}_e(\mathbb{R})$, $p \in \mathbb{N}$, and

$$\widehat{\varphi}_c^{(p)}(t) = \begin{cases} (-1)^{p/2}\mathcal{F}^c(x^p\varphi(x))(t), \ p \text{ is even;} \\ (-1)^{(p-1)/2}\mathcal{F}^c(x^p\varphi(x))(t), \ p \text{ is odd.} \end{cases}$$

Similar remark holds for $\widehat{\varphi}_s(t)$ in $\mathcal{S}_o(\mathbb{R})$.

We have the following result.

**Theorem 2.10.** *Let $f \in \mathcal{S}'_c(\mathbb{R})$ ($f \in \mathcal{S}'_o(\mathbb{R})$) be a quasi-asymptotically bounded at zero with respect to a slowly varying function at infinity L, that is,*

$$|\langle f(\varepsilon x), \varphi(x)\rangle| \leqslant C_\varphi \varepsilon^m L(\varepsilon), \quad \varepsilon \to 0,$$

where $\varphi \in \mathcal{S}_c(\mathbb{R})$ ($\varphi \in \mathcal{S}_o(\mathbb{R})$), and $C_\varphi > 0$ depends of $\varphi$, then the FrFCT (FrFST) for $f$ is a bounded function at 0, i.e., there exists a constant $C > 0$ such that

$$\frac{\left|\langle e^{-ia(\frac{\xi}{\varepsilon})^2/2}\mathcal{F}_\alpha^c f(\frac{\xi}{\varepsilon}), \varphi(\xi)\rangle\right|}{\varepsilon^{m+1}L(\varepsilon)} \leqslant C \quad \left(\frac{\left|\langle e^{-ia(\frac{\xi}{\varepsilon})^2/2}\mathcal{F}_\alpha^s f(\frac{\xi}{\varepsilon}), \varphi(\xi)\rangle\right|}{\varepsilon^{m+1}L(\varepsilon)} \leqslant C\right).$$

*Proof.* Let $f$ be a quasi-asymptotically bounded at zero with respect to $L$. From (2.11) and Remarks 2.8 and 2.9, we have that there exist $k \in \mathbb{N}_0$ and $M > 0$ such that

$$\frac{1}{\varepsilon^{m+1}L(\varepsilon)}\left|\langle e^{-ia(\frac{\xi}{\varepsilon})^2/2}\mathcal{F}_\alpha^c f(\frac{\xi}{\varepsilon}), \varphi(\xi)\rangle\right| = \left|\frac{\sqrt{1-ia}}{b}\frac{L(\varepsilon)}{\varepsilon^m}\langle e^{-ia(\frac{\varepsilon t}{b})^2/2} f(\frac{\varepsilon t}{b}), \widehat{\varphi}_c(t)\rangle\right|$$

$$\leqslant M \cdot C_\alpha \|\widehat{\varphi}_c(t)\|_k < \leqslant M \cdot C_\alpha \sup_{x\in\mathbb{R}, p\leqslant k} (1 + |t|^2)^{k/2} |\widehat{\varphi}_c^{(p)}(t)| < \infty.$$

## 3. Conclusions

In conclusion, in this research we extended recent inquiries into the analysis of integral transforms within specific distributional spaces. Our approach integrates the concept of quasi-asymptotic behavior, as introduced by Zavialov in [27], and we quantify the scaling asymptotic properties of distributions by asymptotic comparisons with Karamata regularly varying functions. In this paper, we characterized the quasi-asymptotic behavior of even (resp., odd) distributions within the context of a Tauberian theorem applied to the FrFCT (resp., FrFST), and by Thrm. 2.3, we established that distributions exhibiting quasi-asymptotic behavior at zero manifest quasi-asymptotic oscillations at infinity through their corresponding FrFCT or FrFST. Additionally, our second result presented by Thrm. 2.10 sheds light on the boundedness of these transforms concerning quasi-asymptotically bounded distributions.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflicts of interest

The authors declare no conflict of interest.

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