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Form-finding of arches subjected to in-plane earthquake loading using graphic statics

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Abstract

This paper presents a form-finding method for arches subjected to self-weight and in-plane horizontal loading due to earthquakes using graphic statics diagrams to perform a thrust line analysis, which can be used for earthquake assessment of arches. The form-finding procedure ensures that a thrust line fits into the arch geometry regardless of the in-plane earthquake direction, and arch forms are defined to resist the design acceleration by guaranteeing a compression-only load path. In research up to this point, thrust line analysis has been implemented by obtaining a suitable shape through a set of geometric manipulations on an initial thrust line in order to obtain the form of the arch by defining the envelope of the thrust lines that need to be contained within the geometry. Instead of manipulating the geometry thrust line, the approach presented in this paper establishes a graphic statics solution to the problem, where structural optimization is conducted by obtaining the force diagram envelope. The force diagram envelope encompasses the force diagrams of the thrust lines that need to be contained within the form of the arch and is, in turn, used to define the diagrams that construct new intrados and extrados of the arch. Because the method explicitly incorporates horizontal forces in a form-finding procedure based on graphic statics, its further exploration might allow for a completely new extension of the possible applications of graphic statics for problems that are based on combined loading cases, where a horizontal force needs to be taken into consideration during the form-finding process.

Keywords: Form finding, earthquake design, graphic statics, force diagram envelope, thrust line

1. Introduction

This paper presents a form-finding method for arches subjected to self-weight and in-plane horizontal loading due to earthquakes using graphic statics diagrams to perform a thrust line analysis.

Graphic statics is a method used in structural design and analysis that solves structures' static equilibrium through geometric constructions. Its key aspect is the relationship between the shape of a structure and the corresponding system of forces, shown in form and force diagrams. This relationship allows designers to observe how changes in one diagram affect the other, giving them direct control over the form and the forces within structural systems. Rankine [1] and Maxwell [2] established the theoretical basis for graphic statics. Rankine first proposed the idea of reciprocity between the form of the structure and the diagram of forces, while Maxwell provided a geometrical procedure for drawing these reciprocal diagrams. Although many have since contributed to refining this framework, graphic statics in its modern form is often credited to Culmann [3], who consolidated mathematical proofs regarding the projective relationship between a funicular polygon and its force polygon, a concept initially proposed by Varignon [4]. Additionally, Cremona played a significant role by formulating graphic statics as a set of guidelines for addressing specific structural challenges [5].

More recently, graphic statics has expanded into three dimensions (3D) [6], [7] allowing for the representation of equilibrium in spatial systems of forces through closed-force polyhedrons [1]. It has been demonstrated that the reciprocal diagrams utilized in graphic statics are planar projections or sections of polyhedral frames and their reciprocal force polyhedrons; thus, 2D graphic statics emerges as a special case within the broader framework of 3D graphic statics [8], [9], [10]. Consequently, 3D graphic statics facilitates the modeling and analysis of spatial structures interactively and presents a fundamentally new perspective and approach to both 2D problems and their applications.

1.1. Form finding and analysis of arches

The shape of arches has garnered significant attention in research, beginning with R. Hooke's findings asserting that the "true mathematical and mechanical form of arches" mirrors the inverse shape of a hanging chain [11]. However, most research has focused on analyzing existing arches and form-finding of arches under vertical gravity loading rather than on appropriately shaping arches for specific loading conditions. Even after renewed interest in arches was triggered later by Heyman's work [12], [13], the research focus remained on the analysis and optimization of arches under vertical loading. In particular, limited literature has addressed determining the suitable form for arches under horizontal loading, such as earthquakes. The optimization of arches subjected to their own weight and lateral seismic action has only recently been explored.

DeJong et al. [14] and Huerta [15] have simulated equivalent earthquake loading by imposing a horizontal load on arches by tilting them. This approach induces a horizontal acceleration and reduces the compressive stresses under gravity. However, as crushing is assumed not to occur, these reduced compressive stresses can be ignored. Thus, an arch will be stable if a thrust line that fits within the masonry is found, combining horizontal acceleration and gravity. Michiels and Adriaenssens [16] used graphic statics to generate arches subjected to gravity and seismic loads based on the geometric manipulation of funicular polygons obtained under statical lateral forces. The thrust line is geometrically manipulated to produce shapes that can resist design acceleration by guaranteeing a compression-only load path.

Thrust line concepts have been further expanded to 3D networks in the thrust network analysis (TNA). This allows for the structural design and analysis of compression-only arches and shells under vertical loading [17]. One of the strengths of the TNA method is that it uses a dual approach, visualizing the forces in the eventual form using graphic statics by employing force network polygons. Marmo and Rosati [18] reformulated and extended the TNA method to include lateral forces. However, they neglected the dual approach and severed the connection between the form and force diagram, thus reducing the applicability of optimization studies.

2. Form-finding methodology

The form-finding method presented in this paper relies on thrust line analysis performed under combined vertical and horizontal loading. The method does not, however, rely on the simulating the horizontal acceleration by tilting the arch, as suggested in earlier work [14], [15], or mirroring the thrust line [16].

The form-finding process starts by selecting an arbitrary catenary arch with a matching span and rising to the desired dimensions. This arch is then divided into discrete blocks, with fewer blocks used for simulating a masonry arch, each representing a voussoir, while a greater number approximates a monolithic structure. Graphic statics is used to generate thrust lines, which, when fitted within the structure, visualize possible compressive 'flow of forces' through the structure. A force polygon is used to establish the thrust line, depicting the equilibrium of forces within the structure (Figure 1). Initially, a load line, "ak," is formulated based on vertical loads (g) and horizontal loads (a·g), since the seismic equivalent static load at each block is assumed to act at its centroid and is directly proportional to its mass. Subsequently, the location of a pole in the force diagram is picked. The force polygon consists of two main components: the load line and the pole. The load line indicates the magnitude and direction of resultant forces at each block centroid, while lines connecting the pole to the load line's vertices define the resultant thrust forces. Once constructed, the force polygon satisfies equilibrium by definition.

Generally, the nodes of the funicular polygon are constructed as intersections of two lines: (i) a line of resultant centroidal force (load line) and (ii) a line of previous thrust force.

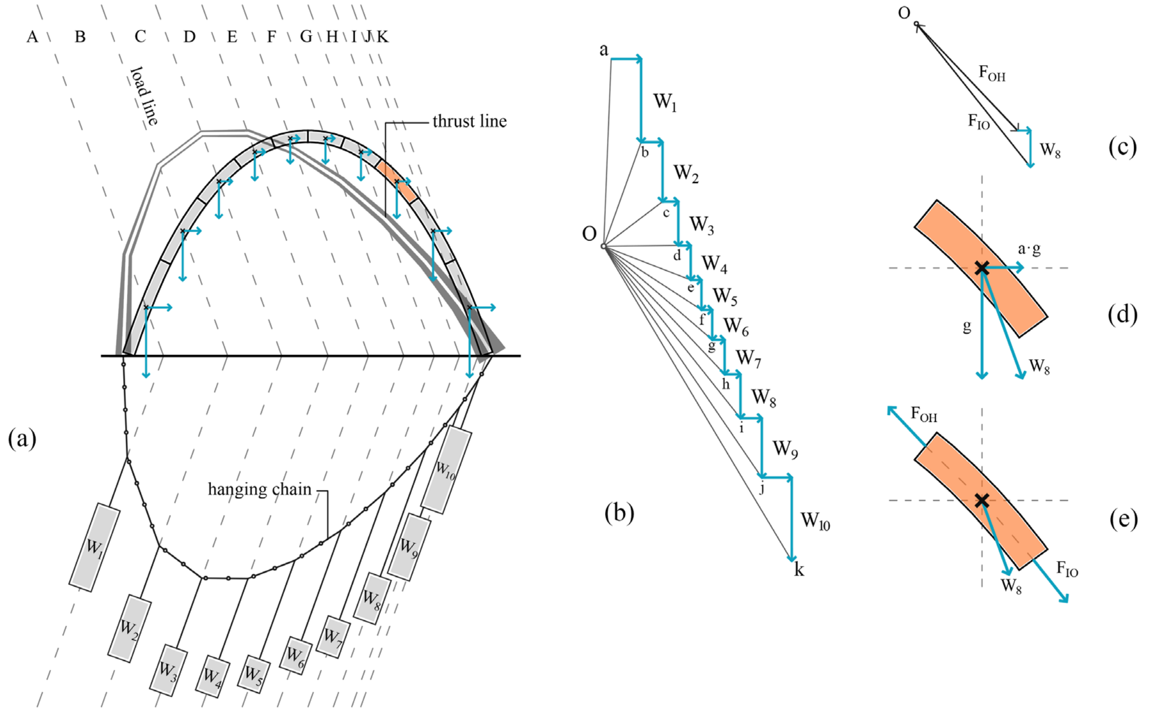

Figure 1: For an initial catenary arch, (a) the thrust line and its equivalent hanging chain constructed using graphic statics; (b) the equilibrium of the system is represented in the force diagram; (c) the equilibrium of one of the segments; (d) resultant centroidal force and, (e) the vectors representing the forces in and on the segment

As the initially defined catenary arch does not accommodate a fitting thrust line, the location of the pole will be adjusted iteratively to obtain the desired rise of the arch. However, the defined thrust line considers only one potential direction of the earthquake. In previous research [16], the thrust line is mirrored along the axis of symmetry of the original arch shape to obtain a second thrust line to account for the other potential direction of the earthquake. Each thrust line is then offset by a distance towards the top and bottom of the initial and mirrored thrust line, leading to four curves, the outline of which always encompasses the initial and mirrored thrust lines. Then, the envelope of these four offset curves (form envelope) is taken as the new shape of the arch. The existing research does not explore the connection between the force diagram and the final form of the arch. In contrast, this research focuses on the geometric manipulation of the initial force diagram to obtain the force diagrams that define extrados and intrados of the form-found arch.

2.1. Diagram of the envelope

In order to clarify the relationship between the force diagram and the final form of the arch, the force diagram is analyzed to isolate the portions that define the upper limit of the arch (the extrados) and the lower limit of the arch (the intrados), as shown in Figure 1(a). The diagram is consequently split into two sections, which are mirrored to account for the alternative direction of the earthquake force. The splitting and mirroring of the force diagram are reflected in the form diagram by introducing a new line of intersection and mirroring of the load lines. Since the mirroring results in symmetrical shapes and the support points are fixed, this new intersection line is located at the midpoint between the supports. This line acts as the mirroring axis for the load lines and represents the shift between the original and mirrored diagram. The extrados and intrados are constructed using the same methodology as the initial thrust line: by intersecting a line of resultant centroidal force (load line) and a line of previous thrust force, with the intersection line serving as the shifting point between the two parts of the diagram.

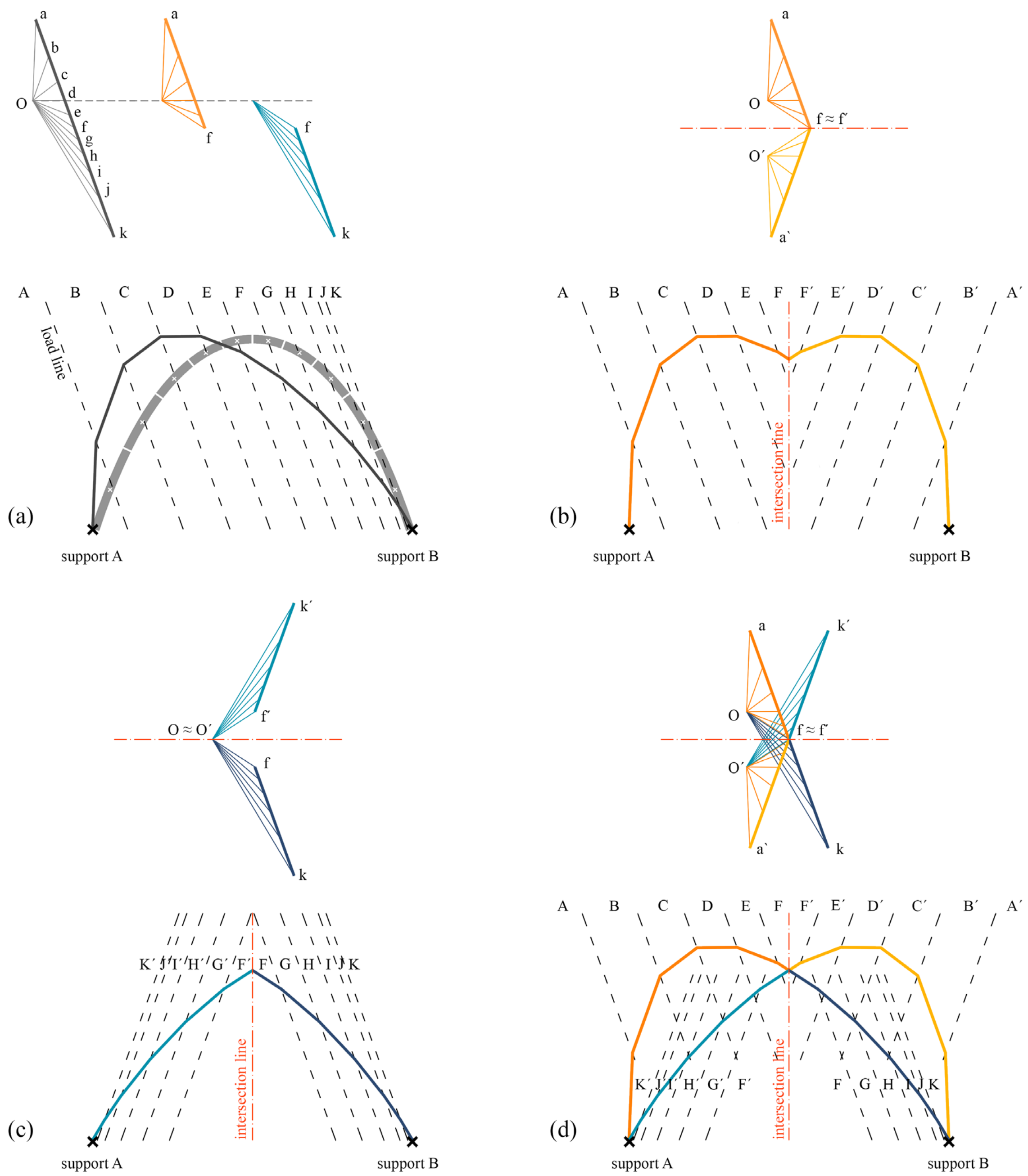

Figure 2: Diagram of the envelope, (a) the initial thrust line and its equivalent force diagram; (b) the extrados of the optimized arch and the relevant diagram sections; (c) the intrados of the optimized arch and the relevant diagram sections; and, (d) the optimized arch form and force diagrams

However, when the location of the thrust line is fixed at the supports, it results in undesirably thick arch sections. This issue can be resolved by displacing the mirrored thrust line horizontally, and the distance between the start and end points of both thrust lines defines the support thickness. This solution results in the arch shape tapering gradually from thick supports to a thin crown. In terms of the force diagram, the extrados and intrados will now be defined by different sections of the original and mirrored force diagram. Since the displacement of the trust lines will result in three intersection points between the original and mirrored thrust line, the force diagram is split into four distinct sections. The fragmentation of the force diagram and the new intersection lines are relative to the displacement of the thrust lines and desired support thickness (Figure 3).

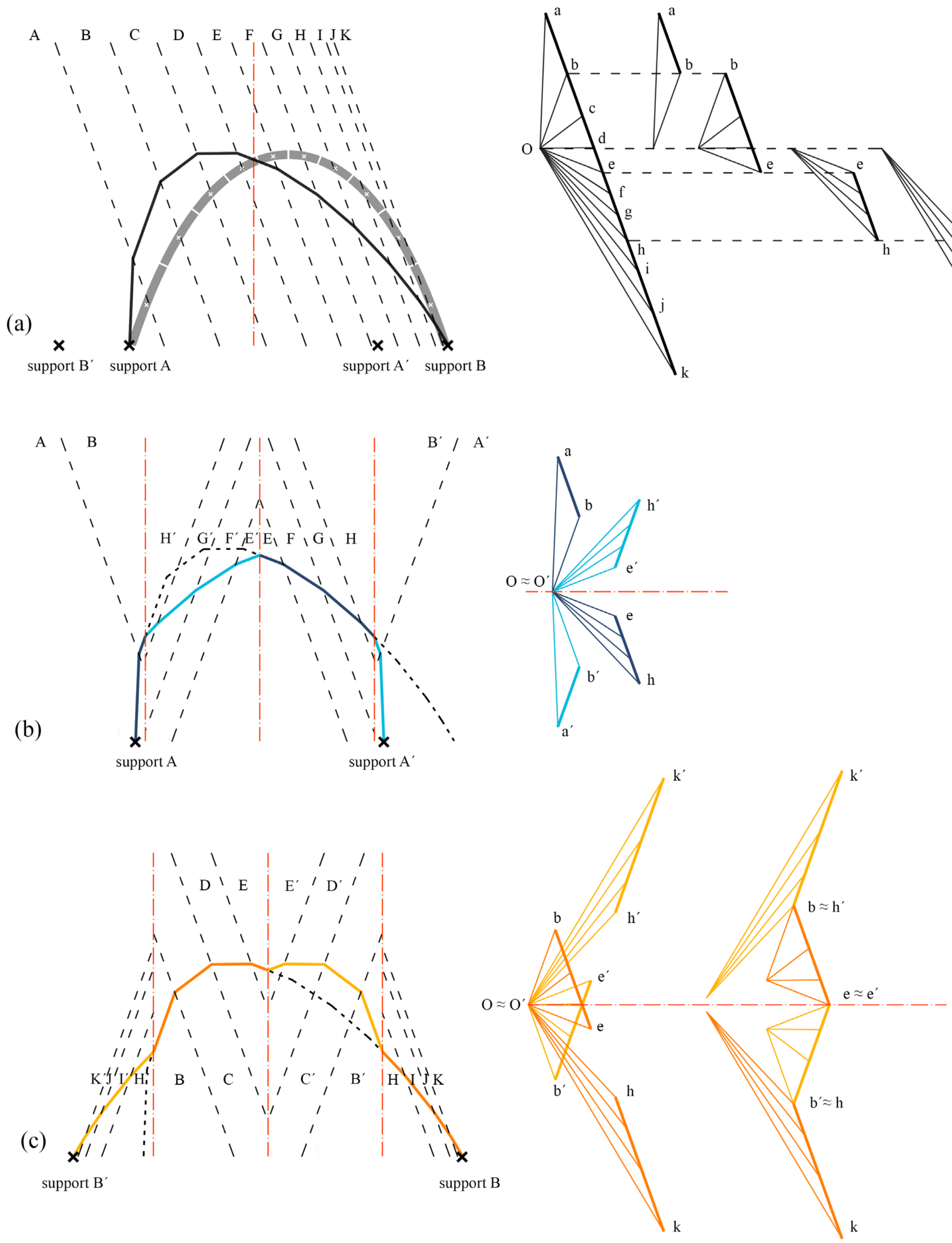

Figure 3: Horizontal displacement of the thrust lines, (a) the initial thrust line and division of the force diagram; (b) the extrados of the optimized arch and the relevant diagram sections; and, (c) the intrados of the optimized arch and the relevant diagram sections;

3. Discussion and further research

The presented methodology clarifies the relationship between the force diagram used to obtain the defining thrust line and the final shape of the arch, creating a basis to further the approach in 3D space. The 2D idea of a thrust line fitting into the arch's form envelope can be expanded to a thrust network in 3D. The analyzed shell will be stable if such a funicular network can fit within the geometry under the combination of self-weight and horizontal acceleration. Thrust Network Analysis (TNA) [17] has extended discretized thrust line analysis to spatial networks, but only for the specific case of gravity loading, using techniques derived from graphic statics. Previous attempts to expand the idea into 3D space for combined vertical and horizontal loading have abandoned the graphic statics approach and used numerical techniques such as particle-spring systems [19], physics simulation engines [20], and dynamic relaxation [21].

However, numerical techniques are limited in their use as design tools because they do not provide a desirable level of control over the process. They do not establish an explicit relationship between the form and forces, making it difficult for a designer to recognize the effect of the form on the distribution of forces. The graphic statics approach differs from such "black-box" structural design tools because the relationship between form and force diagrams is transparent. The magnitude of the force is represented by the length of the corresponding line in the force diagram, and each line in the force diagram is parallel to a line in the form diagram. In this respect, the designer can easily observe the effects of a change in the form diagram due to a modification in the force or vice versa.

This paper relies on the analysis and manipulation of the force diagram in order to define the form of the optimized arch. It isolates the sections of the diagrams that define the extrados and intrados of the arch. In 2D graphic statics, the magnitude of the internal axial force of a member is represented by the length of the corresponding edge in the force diagram. In 3D graphic statics, the areas and orientations of the faces in the polyhedral force diagram represent the directions and the magnitudes of the forces in the corresponding members in the polyhedral form diagram. The next step was to interpret this type of combined loading through a 3D force diagram and attempt to recreate the funicular form. In order to represent the effect of the horizontal seismic force, the horizontal plane of the diagram, which represents the external loads, is inclined at an angle, where the size of the inclination angle represents the magnitude of the horizontal force (Figure 4).

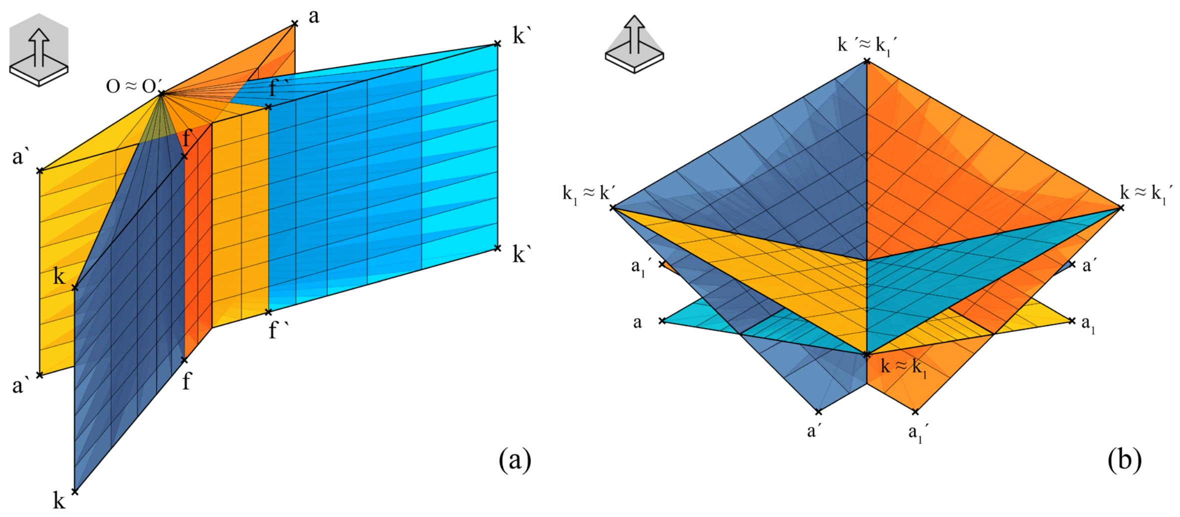

Figure 4: Expansion of the diagrams into 3D space, (a) linear extrusion of the arch diagram to a vault diagram (b) point extrusion of the arch diagram to a shell diagram

4. Conclusion

This paper presents a form-finding technique to obtain efficient shapes of arches subjected to horizontal in-plane accelerations and self-weight. It builds on thrust line analysis, which can be used for a first-order earthquake assessment of arches. The form-finding procedure ensures that a thrust line fits into the arch geometry regardless of the in-plane earthquake direction by obtaining a suitable form of the arch through a set of geometric manipulations of the thrust line force diagram. By taking the envelope of a set of force diagrams under both gravity and horizontal loading, shapes of arches can be obtained that allow a compressive load path to be formed under a designed earthquake load. During the form-finding process, the force diagram is manipulated to obtain various support sizes.

This paper has shown great potential in using interactive thrust line analysis arches and shells. While this paper presents important theoretical groundwork and provides initial insight into the diagram envelope, further work must be conducted to expand the method into 3D space. It has also raised new research questions to be developed and has given possible paths for further exploration and development. Finally, the concept of the thrust line emphasizes the relationship between geometry and structural behavior of arches and shells as a fundamental principle in the design process.

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