Visualization of the nets of type of Zaremba−Halton constructed in generalized number system*

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Abstract. In the present paper a class of two-dimensional nets $\mathcal{Z}^{\kappa,\mu}_{\mathcal{B}_2,\nu}$ of type of Zaremba-Halton constructed in generalized $\mathcal{B}_2-$adic system is introduced. In order to show their very well uniform distribution, we made their visualization with mathematical software *Mathematica*.
In our paper *"On the $(Vil_{\mathcal{B}_2};\alpha;\gamma)-$diaphony of the nets of type of Zaremba - Halton constructed in generalized number system"* (in preparation) we constructed a class $\mathcal{Z}^{\kappa,\mu}_{\mathcal{B}_2,\nu}$ of two-dimensional nets (throughout this paper the term net will denote finite sequence) of type of Zaremba-Halton. Also, the $(Vil_{\mathcal{B}_2};\alpha;\gamma)-$diaphony which is based on using two-dimensional Vilenkin functions constructed in the same $\mathcal{B}_2-$adic system, of the nets of the class $\mathcal{Z}^{\kappa,\mu}_{\mathcal{B}_2,\nu}$ is investigated. The obtained results have theoretical character and treat to the influence of the parameter α to the exact order of the $(Vil_{\mathcal{B}_2};\alpha;\gamma)-$diaphony of the nets from the class $\mathcal{Z}^{\kappa,\mu}_{\mathcal{B}_2,\nu}$.
The purpose of this paper is to present in extended form the visualization of some concrete nets from the class $\mathcal{Z}^{\kappa,\mu}_{\mathcal{B}_2,\nu}$ and to show the distribution of the points of these nets. In this sense the reader can understand the distribution properties of the considered nets. To construct nets of the class $\mathcal{Z}^{\kappa,\mu}_{\mathcal{B}_2,\nu}$ we use the mathematical software *Mathematica*.
Our interest for the importance of these types of sequences is implied from their usage in numerical integration, construction of random number generators e.t.c.

Key words and phrases: Nets of type of Zaremba-Halton, Visualization.

1 Introduction

Let $s \geq 1$ be a fixed integer which will denote the dimension throughout the paper. The functions of some classes of complete orthonormal systems over the

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$s-$dimensional unit cube $[0,1)^s$ are used with a big success as an analytical tool for investigation of the distribution of sequences and nets. Also, these systems stand on the basis of the definitions of some quantitative measures which show the quality of the distribution of sequences and nets. We will remind the definitions of the functions of some complete orthonormal function systems. The first example is the trigonometric system. For an arbitrary integer k the function $e_k : [0,1) \to \mathbb{C}$ is defined as $e_k(x) = e^{2\pi \mathbf{i} kx}, \quad x \in [0,1)$. For an arbitrary vector $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{Z}^s$ the function $e_{\mathbf{k}} : [0,1)^s \to \mathbb{C}$ is defined as $e_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} e_{k_j}(x_j), \ \mathbf{x} = (x_1, \dots, x_s) \in [0,1)^s$. The set $\mathcal{T}_s = \{e_{\mathbf{k}}(\mathbf{x}) : \mathbf{k} \in \mathbb{Z}^s, \ \ \mathbf{x} \in [0,1)^s\}$ is called trigonometric function system.

Following Larcher, Niederreiter and Schmid [9] we will present the concept of the Walsh functions over finite groups. For this purpose let $m \geq 1$ be a given integer and $\{b_1, b_2, \dots, b_m : 1 \leq l \leq m \ b_l \geq 2\}$ be a set of fixed integers. For $1 \leq l \leq m$ let $\mathbb{Z}_{b_l} = \{0, 1, \dots, b_l - 1\}$ the operation $\oplus_{b_l}$ is the addition modulus b_l and $(\mathbb{Z}_{b_l}, \oplus_{b_l})$ is the discrete cyclic group of order b_l.

Let $G = \mathbb{Z}_{b_1} \times \ldots \times \mathbb{Z}_{b_m}$ and the operation $\oplus_G = (\oplus_{b_1}, \dots, \oplus_{b_m})$ is defined as: for arbitrary elements $\mathbf{g}, \mathbf{y} \in G$, where $\mathbf{g} = (g_1, \dots, g_m)$ and $\mathbf{y} = (y_1, \dots, y_m)$, let $\mathbf{g} \oplus_G \mathbf{y} = (g_1 \oplus_{b_1} y_1, \dots, g_m \oplus_{b_m} y_m)$. Then, the couple $(G, \oplus_G)$ is a finite group of order $b = b_1 \dots b_m$.

For arbitrary $\mathbf{g}, \mathbf{y} \in G$ of the above form the character function $\chi_{\mathbf{g}}(\mathbf{y})$ is defined as $\chi_{\mathbf{g}}(\mathbf{y}) = \prod_{l=1}^{m} \exp\left(2\pi \mathbf{i} \frac{g_l y_l}{b_l}\right)$. Let $\varphi : \{0, 1, \dots, b-1\} \to G$ be an arbitrary bijection which satisfies the condition $\varphi(0) = \mathbf{0}$.

For an arbitrary integer $k \geq 0$ and a real $x \in [0,1)$ with the $b-$adic representations $k = \sum_{i=0}^{\nu} k_i b^i$ and $x = \sum_{i=0}^{\infty} x_i b^{-i-1}$, where $k_i, x_i \in \{0, 1, \dots, b-1\}$, $k_\nu \neq 0$ and for infinitely many of $i \ x_i \neq b-1$, the $k-$th Walsh function over the group G with respect to the bijection φ ${}_{G,\varphi}wal_k : [0,1) \to \mathbb{C}$ is defined as ${}_{G,\varphi}wal_k(x) = \prod_{i=0}^{\nu} \chi_{\varphi(k_i)}(\varphi(x_i))$.

Let us denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For an arbitrary vector $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$ the function ${}_{G,\varphi}wal_{\mathbf{k}} : [0,1)^s \to \mathbb{C}$ is defined as ${}_{G,\varphi}wal_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} {}_{G,\varphi}wal_{k_j}(x_j), \ \ \mathbf{x} = (x_1, \dots, x_s) \in [0,1)^s$. The set $\mathcal{W}_{G,\varphi} = \{{}_{G,\varphi}wal_{\mathbf{k}}(\mathbf{x}) : \mathbf{k} \in \mathbb{N}_0^s, \mathbf{x} \in [0,1)^s\}$ is called a system of Walsh functions over the group G with respect to the bijection φ.

In the case when $m = 1$, $G = \mathbb{Z}_b$ and $\varphi = \mathrm{id}$ is the identity between the sets $\{0, 1, \dots, b-1\}$ and $\mathbb{Z}_b$ the obtained function system $\mathcal{W}_{\mathbb{Z}_b, \mathrm{id}}$ is the system $\mathcal{W}(b)$ of the Walsh functions of order b which was proposed by Chrestenson [1]. If $b = 2$ then, the system $\mathcal{W}_{\mathbb{Z}_2, \mathrm{id}}$ is the original system $\mathcal{W}(2)$ of the Walsh [13] functions.

The numerical measures for the irregularity of the distribution of sequences and nets in $[0,1)^s$ give us the order of the inevitable deviation of a concrete distribution from the ideal distribution. Generally, they are different kinds of the discrepancy and the diaphony. So, let $\xi_N = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1}\}$ be an arbitrary net of $N \geq 1$ points in $[0,1)^s$.

For an arbitrary vector $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$ let us denote $[\mathbf{0}, \mathbf{x}) = [0, x_1) \times \ldots \times [0, x_s)$ and $A(\xi_N; [\mathbf{0}, \mathbf{x})) = \#\{\mathbf{x}_n : 0 \leq n \leq N-1, \mathbf{x}_n \in [\mathbf{0}, \mathbf{x})\}$. The L_2-discrepancy of the net ξ_N is defined as

$$T(\xi_N) = \left(\int_{[0,1]^s} \left| \frac{A(\xi_N; [\mathbf{0}, \mathbf{x}))}{N} - \prod_{j=1}^{s} x_j \right|^2 dx_1 \ldots dx_s \right)^{\frac{1}{2}}.$$

In 1976 Zinterhof [14] introduced the notion of the so - called classical diaphony. So, the diaphony of the net ξ_N is defined as

$$F(\mathcal{T}_s; \xi_N) = \left(\sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \{0\}} R^{-2}(\mathbf{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} e_{\mathbf{k}}(\mathbf{x}_n) \right|^2 \right)^{\frac{1}{2}},$$

where for each vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{Z}^s$ the coefficient $R(\mathbf{k}) = \prod_{j=1}^{s} R(k_j)$ and for an arbitrary integer k

$$R(k) = \begin{cases} 1, & \text{if } k = 0, \\ |k|, & \text{if } k \neq 0. \end{cases}$$

In order to study sequences and nets constructed in $b-$adic system, in the last twenty years, some new versions of the diaphony was defined. These kinds of the diaphony are based on using complete orthonormal function systems constructed also in base b. In 2001 Grozdanov and Stoilova [5] used the system $\mathcal{W}(b)$ of the Walsh function to introduce the concept the so-called $b-$adic diaphony. In 2003 Grozdanov, Nikolova and Stoilova [4] used the system $\mathcal{W}_{G,\varphi}$ of the Walsh functions over the group G with respect to the bijection φ to introduce the concept of the so-called generalized $b-$adic diaphony. So, the generalized $b-$adic diaphony of the net ξ_N is defined as

$$F(\mathcal{W}_{G,\varphi}; \xi_N) = \left(\frac{1}{(b+1)^s - 1} \sum_{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\}} \rho(\mathbf{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_{G,\varphi}wal_{\mathbf{k}}(\mathbf{x}_n) \right|^2 \right)^{\frac{1}{2}},$$

where for a vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ the coefficient $\rho(\mathbf{k}) = \prod_{j=1}^{s} \rho(k_j)$ and for an arbitrary non-negative integer k

$$\rho(k) = \begin{cases} 1, & \text{if } k = 0, \\ b^{-2g}, & \text{if } b^g \leq k \leq b^{g+1}, \quad g \geq 0, \quad g \in \mathbb{Z}. \end{cases}$$

In 1954 Roth [11] obtained a general lower bound of the L_2-discrepancy of arbitrary net. He prove that for any net ξ_N composed of N points in $[0,1)^s$ the lower bound

$$T(\xi_N) > C(s)\frac{(\log N)^{\frac{s-1}{2}}}{N} \tag{1}$$

holds, where $C(s)$ is a positive constant depending only on the dimension s.

In 1986 Proinov [10] obtained a general lower bound of the diaphony of arbitrary net. So, for any net ξ_N composed of N points in $[0,1)^s$ the lower bound

$$F(\mathcal{T}_s;\xi_N) > \alpha(s)\frac{(\log N)^{\frac{s-1}{2}}}{N} \tag{2}$$

holds, where $\alpha(s)$ is a positive constant depending only on the dimension s.

Cristea and Pillichshammer [2] obtained a general lower bound of the $b-$adic diaphony of arbitrary net. So, for any net ξ_N composed of N points in $[0,1)^s$ the lower bound

$$F(\mathcal{W}(b);\xi_N) \geq C(b,s)\frac{(\log N)^{\frac{s-1}{2}}}{N} \tag{3}$$

holds, where $C(b,s)$ is a positive constant depending on the base b and the dimension s.

About the exactness of the lower bound (1) of the L_2-discrepancy of two - dimensional nets the following results are obtained. Let $b \geq 2$ and $\nu > 0$ be fixed integers.

Following Van der Corput [12] and Halton [6] for an arbitrary integer i such that $0 \leq i \leq b^\nu - 1$ with the $b-$adic representation $i = \sum_{j=0}^{\nu-1} i_j b^j$ we put $p_{b,\nu}(i) = \sum_{j=0}^{\nu-1} i_j b^{-j-1}$. Let us denote $\eta_{b,\nu}(i) = \frac{i}{b^\nu}$ and to construct the net $R_{b,\nu} = \{(\eta_{b,\nu}(i), p_{b,\nu}(i)) : 0 \leq i \leq b^\nu - 1\}$. The net $R_{2,\nu}$ was introduced and studied by Roth [11]. Hammersley [8] used sequences of Van der Corput$-$Halton constructed in different bases to construct a class of $s-$dimensional nets.

Roth proved that the L_2-discrepancy $T(R_{2,\nu})$ of the net $R_{2,\nu}$ have an exact order $\mathcal{O}\left(\frac{\log N}{N}\right)$, where $N = 2^\nu$. According to the lower bound (1) the order $\mathcal{O}\left(\frac{\log N}{N}\right)$ is not the best possible.

There exist two important techniques to improve the order of the L_2-discrepancy of arbitrary net. They are a digital shift and a symmetrization of the points of the net. The first approach was realized in 1969 by Halton and Zaremba [7]. They used the original net of Van der Corput $\{p_{2,\nu}(i) = 0, i_0 i_1 \dots i_{\nu-1} : i_j \in \{0,1\},\ 0 \leq j \leq \nu - 1\}$ and changed the digits i_j that stay on the even positions with the digits $1 - i_j$. In this way the net $\{z_{2,\nu}(i) = 0,(1-i_0)i_1(1-i_2)\dots : i_j \in \{0,1\},\ 0 \leq j \leq \nu - 1\}$ is obtained. The net $Z_{2,\nu} = \{(\eta_{2,\nu}(i), z_{2,\nu}(i)) : 0 \leq i \leq 2^\nu - 1\}$ which now usually is called net of

Zaremba−Halton is constructed. It is shown that the L_2-discrepancy of the net $Z_{2,\nu}$ has an order $\mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right)$, $N = 2^\nu$, which of course is the best possible.

In 1998 Xiao proved the exactness of the lower bound (2) of the diaphony for dimension $s = 2$. It is shown that the diaphony of the nets of Roth $R_{b,\nu}$ and Zaremba−Halton $Z_{b,\nu}$, both constructed in arbitrary base $b \geq 2$, have an exact order $\mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right)$, $N = b^\nu$.

Grozdanov and Stiolova [5] proved the exactness of the lower bound (3) of the $b-$adic diaphony for dimension $s = 2$. They proved that the $b-$adic diaphony of the nets $R_{b,\nu}$ and $Z_{b,\nu}$ have an exact order $\mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right)$, $N = b^\nu$.

Grozdanov [3] constructed the so-called net of Zaremba−Halton ${}_{G,\varphi}Z^{p,q}_{b,\nu}$ over the group G with respect to the bijection φ and obtained the exact order $\mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right)$ of the generalized $b-$adic diaphony of the nets $R_{b,\nu}$ and ${}_{G,\varphi}Z^{p,q}_{b,\nu}$.

2 The nets of type of Zaremba-Halton constructed in generalized system

We will remind the concept of the so-called generalized number system or a system with variable bases. Let the sequence of integers $B = \{b_0, b_1, b_2, \ldots : b_i \geq 2 \text{ for } i \geq 0\}$ be given. The so-called generalized powers are defined by the next recursive equalities. We put $B_0 = 1$ and for $j \geq 0$ define $B_{j+1} = B_j.b_j$. Then, an arbitrary integer $k \geq 0$ and a real $x \in [0, 1)$ in the generalized B-adic number system can be represented of the form

$$k = \sum_{i=0}^{\nu} k_i B_i \quad \text{and} \quad x = \sum_{i=0}^{\infty} \frac{x_i}{B_{i+1}},$$

where for $i \geq 0$, $k_i,\ x_i \in \{0, 1, \ldots, b_i - 1\}$, $k_\nu \neq 0$ and for infinitely many i we have $x_i \neq b_i - 1$.

For arbitrary reals $x, y \in [0, 1)$ which in the $B-$adic system have the representations of the form $x = \sum_{i=0}^{\infty} \frac{x_i}{B_{i+1}}$ and $y = \sum_{i=0}^{\infty} \frac{y_i}{B_{i+1}}$, where for $i \geq 0$ $x_i, y_i \in \{0, 1, \ldots, b_i - 1\}$ and for infinitely many i we have $x_i, y_i \neq b_i - 1$, we define the operation

$$x \oplus_B^{[0,1)} y = \left(\sum_{i=0}^{\infty} \frac{x_i + y_i (\text{mod } b_i)}{B_{i+1}} \right) (\text{mod } 1).$$

In the next definition we will present the concept of the sequence of Van der Corput constructed in the generalized $B-$adic system.

Definition 1 *For an arbitrary integer $i \geq 0$ which in the generalized $B-$adic system has the representation $i = i_m B_m + i_{m-1} B_{m-1} + \cdots + i_0 B_0$, where for $0 \leq j \leq m$ $i_j \in \{0, 1, \ldots, b_j - 1\}$ and $i_m \neq 0$, we put*

$$p_B(i) = \frac{i_0}{B_1} + \frac{i_1}{B_2} + \ldots + \frac{i_m}{B_{m+1}}.$$

The sequence $\omega_B = (p_B(i))_{i \geq 0}$ is called a sequence of Van der Corput constructed in the generalized system B.

The sequence ω_B was investigated by many several authors and used for different purposes.

Now we explain the constructive principle of two-dimensional nets (introduced in our mentioned paper) of the type of Zaremba-Halton constructed in generalized system.

For this purpose let $B_1 = \{b_0^{(1)}, b_1^{(1)}, \ldots, b_{\nu-1}^{(1)}, b_\nu^{(1)}, b_{\nu+1}^{(1)}, \ldots : b_j^{(1)} \geq 2 \text{ for } j \geq 0\}$ be a given sequence of bases. By using the sequence B_1 we will define the sequence $B_2 = \{b_0^{(2)}, b_1^{(2)}, \ldots, b_{\nu-1}^{(2)}, b_\nu^{(2)}, b_{\nu+1}^{(2)}, \ldots : b_j^{(2)} \geq 2 \text{ for } j \geq 0\}$ in the following manner. We put $b_0^{(2)} = b_{\nu-1}^{(1)}$, $b_1^{(2)} = b_{\nu-2}^{(1)}, \ldots, b_{\nu-1}^{(2)} = b_0^{(1)}$ and the bases $b_\nu^{(2)}, b_{\nu+1}^{(2)}, \ldots$ can be chosen arbitrary. Let us denote $\mathcal{B}_2 = (B_1, B_2)$.

Let us assume that the sequences B_1 and B_2 of bases are *limited from above*, i. e. there exists a constant $M \geq 2$ such that for each $j \geq 0$ and $\tau = 1, 2$ we have $b_j^{(\tau)} \leq M$.

Let $\nu \geq 1$ be an arbitrary and fixed integer. We have that $B_\nu^{(1)} = b_0^{(1)}.b_1^{(1)} \ldots b_{\nu-1}^{(1)}$ and $B_\nu^{(2)} = b_0^{(2)}.b_1^{(2)} \ldots b_{\nu-1}^{(2)} = b_{\nu-1}^{(1)}.b_{\nu-2}^{(1)} \ldots b_0^{(1)}$, so $B_\nu^{(1)} = B_\nu^{(2)}$. Let us denote $B_\nu = B_\nu^{(1)} = B_\nu^{(2)}$. For $0 \leq i \leq B_\nu - 1$ let us define the quantity $\eta_\nu(i) = \dfrac{i}{B_\nu}$.

Let $\kappa = 0, \kappa_{\nu-1}\kappa_{\nu-1} \ldots \kappa_0$, where for $0 \leq j \leq \nu - 1$ $\kappa_{\nu-1-j} \in \{0, 1, \ldots, b_j^{(1)} - 1\}$ be a fixed B_1-adic rational number.

Let $\mu = 0, \mu_0\mu_1 \ldots \mu_{\nu-1}$, where for $0 \leq j \leq \nu - 1$ $\mu_j \in \{0, 1, \ldots, b_j^{(2)} - 1\}$ be a fixed B_2-adic rational number.

The different choice of the parameters κ and μ permits us to obtain a class of two-dimensional nets with similar construction and distribution properties of the points in the plane.

For an arbitrary i such that $0 \leq i \leq B_\nu - 1$ let us denote

$$\eta_\nu^\kappa(i) = \eta_\nu(i) \oplus_{B_1}^{[0,1)} \kappa \quad \text{and} \quad z_\nu^\mu(i) = p_{B_2}(i) \oplus_{B_2}^{[0,1)} \mu.$$

Definition 2 *Let $\nu \geq 1$ be an arbitrary and fixed integer. Let κ and μ be as above. The two-dimensional net*

$$Z_{\mathcal{B}_2,\nu}^{\kappa,\mu} = \{(\eta_\nu^\kappa(i), z_\nu^\mu(i)) : 0 \leq i \leq B_\nu - 1\}$$

we will call a net of type of Zaremba-Halton constructed in the generalized $\mathcal{B}_2-$adic system.

In our mentioned paper we investigated and proved "very well" uniform distribution of these nets.

Our interest for the importance of these types of sequences is implied from their usage in numerical integration, construction of random number generators e.t.c.

3 Visualization results for nets of type of Zaremba-Halton

We will use the above mathematical model to present a version of the program in *Mathematica* which can construct arbitrary net of type of Zaremba - Halton, for the chosen parameters.

In our work we will confine only to present a visualization of some concrete nets from the class $Z_{\mathcal{B}_2,\nu}^{\kappa,\mu}$ and to show the distribution of the points of these nets.

Some theoretical investigations of the nets from the class $Z_{\mathcal{B}_2,\nu}^{\kappa,\mu}$ have been realized in other paper of the authors.

Our code of the program, for some choices of parameters, is presented below:

```
(* parameters *)
nu = 3; mi = {2, 1, 3}; ni = {1, 0, 1};
b2[0] = 5; b2[1] = 2; b2[2] = 3; b2v = {5, 2, 3};
b1[0] = 3; b1[1] = 2; b1[2] = 5; b1v = {3, 2, 5};
B1[0] = B2[0] = 1; niza = {};
(* end parameters *)

Do[
    B1[j] = B1[j - 1]*b1[j - 1];
    B2[j] = B2[j - 1]*b2[j - 1],
{j, nu}]

For[i = 0, i < b2[2], i++,
        For[j = 0, j < b2[1], j++,
                For[k = 0, k < b2[0], k++, pc = {i, j, k};

                        p = Reverse[pc];
                        z = Mod[mi+p, b2v];
                        zv = Sum[z[[l]]/B2[l],{l, nu}];
                        et = Mod[ni + pc, b1v];
                        etv = Sum[et[[l]]/B1[l],{l, nu}];
                        AppendTo[niza,{etv,zv}]

                ]]]
```

We illustrate the work of the code by some examples.

In the first example we make the following choice of the parameters: $\nu = 3$, $b_0^{(1)} = 3$, $b_1^{(1)} = 2$, $b_2^{(1)} = 5$, $b_0^{(2)} = 5$, $b_1^{(2)} = 2$, $b_2^{(2)} = 3$, $\kappa = 0.101$ and $\mu = 0.213$. The distribution of the points of this net is given in Fig. 1.

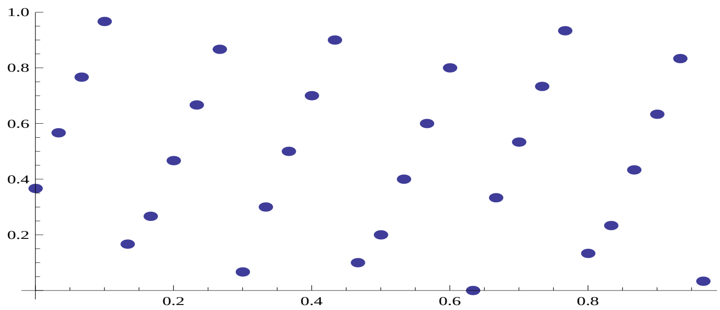

Fig. 1. $b_0^{(1)} = 3$, $b_1^{(1)} = 2$, $b_2^{(1)} = 5$, $\kappa = 0.101$ and $\mu = 0.213$.

The points of the net $Z_{\mathcal{B}_2,\nu}^{\kappa,\mu}$ are:

$$Z_{\mathcal{B}_2,\nu}^{\kappa,\mu} = \left\{ \left(\frac{11}{30}, \frac{15}{30}\right), \left(\frac{12}{30}, \frac{21}{30}\right), \left(\frac{13}{30}, \frac{27}{30}\right), \left(\frac{14}{30}, \frac{3}{30}\right), \left(\frac{10}{30}, \frac{9}{30}\right), \left(\frac{16}{30}, \frac{12}{30}\right), \right.$$
$$\left(\frac{17}{30}, \frac{18}{30}\right), \left(\frac{18}{30}, \frac{24}{30}\right), \left(\frac{19}{30}, \frac{0}{30}\right), \left(\frac{15}{30}, \frac{6}{30}\right), \left(\frac{21}{30}, \frac{16}{30}\right), \left(\frac{22}{30}, \frac{22}{30}\right),$$
$$\left(\frac{23}{30}, \frac{28}{30}\right), \left(\frac{24}{30}, \frac{4}{30}\right), \left(\frac{20}{30}, \frac{10}{30}\right), \left(\frac{26}{30}, \frac{13}{30}\right), \left(\frac{27}{30}, \frac{19}{30}\right), \left(\frac{28}{30}, \frac{25}{30}\right), \left(\frac{29}{30}, \frac{1}{30}\right),$$
$$\left(\frac{25}{30}, \frac{7}{30}\right), \left(\frac{1}{30}, \frac{17}{30}\right), \left(\frac{2}{30}, \frac{23}{30}\right), \left(\frac{3}{30}, \frac{29}{30}\right), \left(\frac{4}{30}, \frac{5}{30}\right), \left(\frac{0}{30}, \frac{11}{30}\right),$$
$$\left. \left(\frac{6}{30}, \frac{14}{30}\right), \left(\frac{7}{30}, \frac{20}{30}\right), \left(\frac{8}{30}, \frac{26}{30}\right), \left(\frac{9}{30}, \frac{2}{30}\right), \left(\frac{5}{30}, \frac{8}{30}\right) \right\}.$$

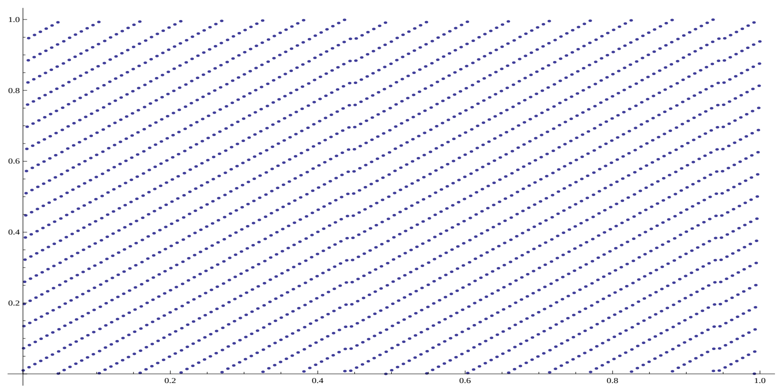

Fig. 2. $b_0^{(1)} = 2$, $b_1^{(1)} = 9$, $b_2^{(1)} = 7$, $b_3^{(1)} = 16$, $\kappa = 0.180F$ and $\mu = 0.F101$.

In the second example we take $\nu = 4$, $b_0^{(1)} = 2$, $b_1^{(1)} = 9$, $b_2^{(1)} = 7$, $b_3^{(1)} = 16$, $b_0^{(2)} = 16$, $b_1^{(2)} = 7$, $b_2^{(2)} = 9$,$b_3^{(2)} = 2$, $\kappa = 0.180F$ and $\mu = 0.F101$, where F is the symbol for the number 15 in the number system with base 16.

The distribution of the points of the second net is given in Fig. 2.

In the third example (see Fig.3) we make the following choice of the parameters: $\nu = 4$, $b_0^{(1)} = 2$, $b_1^{(1)} = 3$, $b_2^{(1)} = 7$, $b_3^{(1)} = 11$.

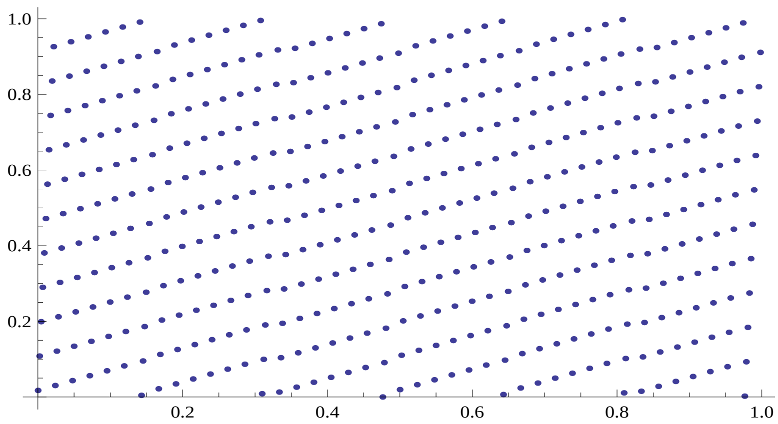

Fig. 3. $b_0^{(1)} = 2$, $b_1^{(1)} = 3$, $b_2^{(1)} = 7$, $b_3^{(1)} = 11$.

In the next example (see Fig. 4) we make the following choice of the parameters: $\nu = 6$, $b_0^{(1)} = 2$, $b_1^{(1)} = 3$, $b_2^{(1)} = 2$, $b_3^{(1)} = 5$, $b_4^{(1)} = 5$, $b_5^{(1)} = 3$.

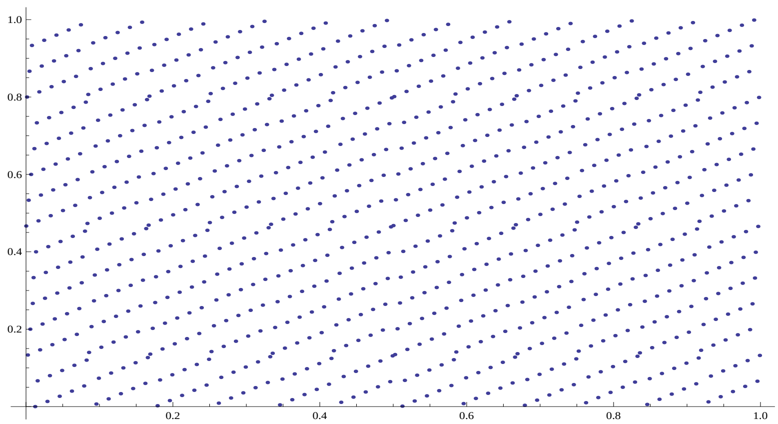

Fig. 4. $b_0^{(1)} = 2$, $b_1^{(1)} = 3$, $b_2^{(1)} = 2$, $b_3^{(1)} = 5$, $b_4^{(1)} = 5$, $b_5^{(1)} = 3$.

Some more examples with $\nu = 6$, are presented on the Fig. 5 ($b_0^{(1)} = 2$, $b_1^{(1)} = 3$, $b_2^{(1)} = 4$, $b_3^{(1)} = 5$, $b_4^{(1)} = 6$, $b_5^{(1)} = 8$) and Fig. 6 ($b_0^{(1)} = 13$, $b_1^{(1)} = 11$, $b_2^{(1)} = 7$, $b_3^{(1)} = 5$, $b_4^{(1)} = 3$, $b_5^{(1)} = 2$).

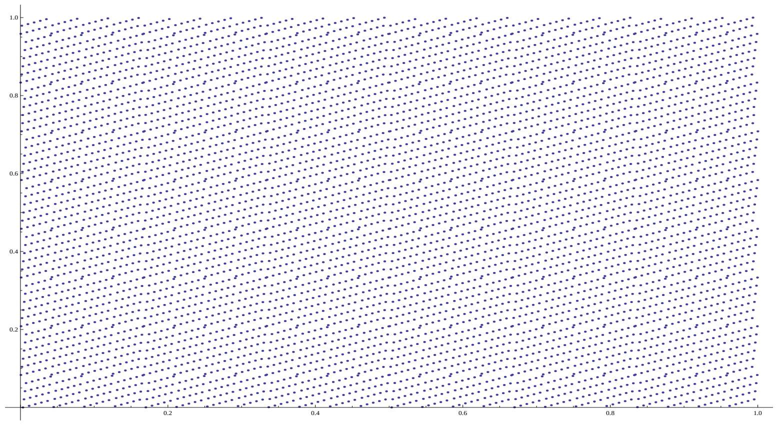

Fig. 5. $b_0^{(1)} = 2$, $b_1^{(1)} = 3$, $b_2^{(1)} = 4$, $b_3^{(1)} = 5$, $b_4^{(1)} = 6$, $b_5^{(1)} = 8$.

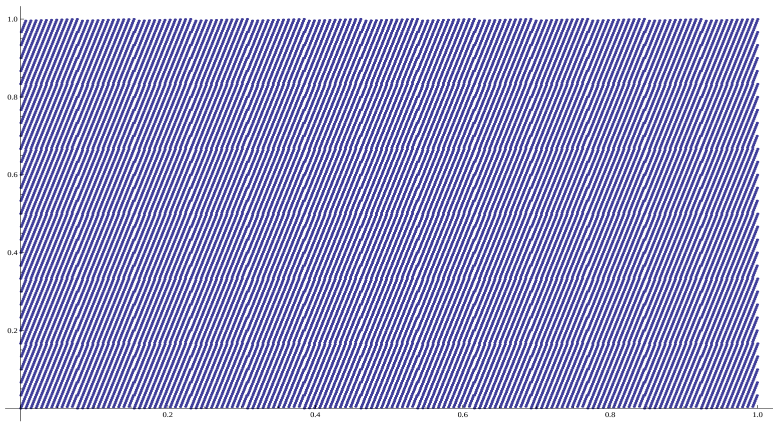

Fig. 6. $b_0^{(1)} = 13$, $b_1^{(1)} = 11$, $b_2^{(1)} = 7$, $b_3^{(1)} = 5$, $b_4^{(1)} = 3$, $b_5^{(1)} = 2$.

4 Conclusion

We made visualizations with mathematical software *Mathematica* of the two-dimensional nets $\mathcal{Z}^{\kappa,\mu}_{\mathcal{B}_2,\nu}$ of type of Zaremba-Halton constructed in generalized $\mathcal{B}_2-$adic system.

We show some examples with different number of points, different bases and different shift - vectors.

All of the realized experiments with graphical representations, where the number of points in the nets was between 24 and 2 000 000, very well confirm the theoretical results, about uniform distribution of the points in the introduced net, obtained in our previous mentioned paper.

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