RELIABILITY FUNCTIONS FOR MULTI-STATE SYSTEMS WITH EQUAL GRADUATE FAILURES

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Abstract: In some systems failures do not occur suddenly, but they are a result of accumulation of a sequence of many gradual failures. In this paper we consider systems with equal gradual failures in which it is supposed that all failures can be repaired with same probability. We present a reliability model for this type of systems, calculate the reliability functions and give a simulation with Matematica.

Keywords: reliability, graduate failures, failure time, transition time, repairs time, multi-state system

1 Introduction

There are systems in which complete failure of components occurs as a sequence of gradual failures. Such failures may change quality of a component, but usually a component still works. Combinations of gradual failures of components may change quality of the whole system and consequently system may have different states. Also the combinations of gradual failures may lead to complete system failure. We are focus on the systems which can be repaired to its perfect state, before total failure of it.

In this paper we consider probabilistic aspect of this type of systems and propose a model in which components have equal probability for one level failure. In section 2 we consider reliability of a unit component. Section 3 gives possibilities for approximate explicit calculation of density function of the time to first entry in some state. The systems with several components are considered in section 4. Here we give reliability function for series systems and parallel systems in the case when intensity of failure is change on that way so the sum of the intensities of failure of all components keeps constant. In section 5 we give some ways for estimating intensities of repairs and failures in different cases. At the end we present some concrete results and experiments.

2 A model with one component

We consider a reliability model of a simple unit with graduate failures. Let the state space of the unit be represent with the set $E=\{0,1,2, ..., n\}$. This numbers indicate a possible level of gradual failure of the unit, and greater number represents greater working level of the unit. So *n* means perfect level and 0 means a total failure. Graduate failure means that only possible downfall of the unit is from state *i* to state *i-1*. The unit can be repaired in such a way that after any repair the unit returns back to the initial state, which is the perfect state.

The behaviour of the unit is represent by a Markov Process $X=\{X(t):t\geq 0\}$ with discrete state space E and transition intensities λ for failure and μ for repair. So, the one level down transition time, (the time needed to fall from state i to state i-1) has an exponential distribution with parameter λ, and repair time (time needed to get from state i to state n) has exponential distribution with parameter μ. Let T_k is the random variable that represents the time to the first entry at level k. Then the reliability function for level k is:

$$R_k(t)=P(T_k>t) \tag{1}$$

We would like to find density function for this random variable T_k.

We consider a unit that falls down from state n to state k (n>k) with r repairs, $r\in\mathbf{N}$. This can be done such that the state of the unit continuously falls down to state $p_i\in\{k+1, k+2, ..., n-1\}$ for i=1, ..., r and the unit is repaired and the last time state of the unit continuously falls down to the state k. Let m_1 of these r states be equal to n−1, m_2 to n−2 est... We define a vector $(m_1, m_2, ..., m_{n-k-1})$ where $m_1+m_2+ ...+m_{n-k-1}=r$ and m_i means that from all r repairs, m_i are repairs from the state n-i to the perfect state. Let us fix the vector $(m_1, m_2, ..., m_{n-k-1})$. There are

$$\frac{r!}{m_1!m_2!...m_{n-k-1}!} \tag{2}$$

ways for received this vector. Then the probability to get such a vector is:

$$\left(\frac{\mu}{\lambda+\mu}\right)^r\left(\frac{\lambda}{\lambda+\mu}\right)^{n-k-1+\sum_{i=2}^{n-k-1}(i-1)m_i} \tag{3}$$

So, the density function of the transition time from initial state n to state k given the vector $(m_1, m_2, ..., m_{n-k-1})$ is:

$$h(\lambda,\ r+1,\ \lambda+\mu,\ n-k-1+\sum_{i=1}^{n-k-1} im_i,\ t) \tag{4}$$

where

$$h(\lambda, k_1, \mu, k_2, t)=\int_0^t \Gamma(k_1,\lambda,t-x)\Gamma(k_2,\mu,x)dx \tag{5}$$

$$\Gamma(l,k,t)=\int_0^t \frac{l^k t^{k-1} Exp(-lt)}{\Gamma(k)}$$

Let us define a set A_r by:

$$A_r=\{(m_1, m_2, ..., m_{n-k-1})\,|\, m_1+m_2+ ...+m_{n-k-1}=r\} \tag{6}$$

Now the density function of T_k is:

$f_{k,n}(\lambda,\mu,t)=$

$$\sum_{r=0}^{\infty}\left(\frac{\mu}{\lambda+\mu}\right)^{r}\sum_{m\in A_r}\left(\frac{\lambda}{\lambda+\mu}\right)^{n-k-1+\sum_{n=2}^{n-k-1}(i-1)m_i}\frac{r!}{\prod_{i=1}^{n-k-1}m_i!}h(\lambda,r+1,\mu+\lambda,n-k-1+\sum_{i=1}^{n-k-1}im_i,t) \quad (7)$$

and reliability function of T_k is:

$$R_{k,n}(t)=\int_t^{\infty}\mathrm{f}_{\mathrm{k,n}}(\lambda,\mu,t)dt \quad (8)$$

Density function of the total failure time is found for k=0 and it is:

$f_n(\lambda,\mu,t)=$

$$\sum_{r=0}^{\infty}\left(\frac{\mu}{\lambda+\mu}\right)^{r}\sum_{m\in A_r}\left(\frac{\lambda}{\lambda+\mu}\right)^{n-1+\sum_{i=2}^{n-1}(i-1)m_i}\frac{r!}{\prod_{i=1}^{n-1}m_i!}h(\lambda,r+1,\mu+\lambda,n-1+\sum_{i=1}^{n-1}im_i,t) \quad (9)$$

for reliability function is obtain:

$$R_n(t)=\int_t^{\infty}\mathrm{f}_{\mathrm{n}}(\lambda,\mu,s)ds \quad (10)$$

3 Numerical calculation for density function

We see that the density function of T_k (9), is an infinite convergent series, and for large r the members of this sum are sufficiently small. So it is possible to approximate this infinite series with a finite sum ignoring all members in that series after some r_0 with an error ε. Let us select a positive arbitrary ε. We would like to find a function such that the difference between that function and the density function f_k is smaller than ε. For this reason we will calculate the probability for obtaining vector (m_1, m_2, ..., m_{n-k-1}) for which the sum:

$$\sum_{i=1}^{n-k-1}m_i \geq r_0 \quad (11)$$

This probability is equal to:

$$P\left(\sum_{i=1}^{n-k-1}m_i \geq r_0\right)=\sum_{r=r_0}^{\infty}\left(\frac{\mu}{\lambda+\mu}\right)^{r}\sum_{m\in A_r}\left(\frac{\lambda}{\lambda+\mu}\right)^{n-k-1+\sum_{n=2}^{n-k-1}(i-1)m_i}\frac{r!}{\prod_{i=1}^{n-k-1}m_i!}=$$

$$=\left(\frac{\lambda}{\lambda+\mu}\right)^{n-k-1}\sum_{r=r_0}^{\infty}\left(\frac{\mu}{\lambda+\mu}\right)^{r}\left(\sum_{j=0}^{n-k-2}\left(\frac{\lambda}{\lambda+\mu}\right)^{j}\right)^{r}$$

$$=\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{n-k-1}\right)^{r_0} \tag{12}$$

So we choose r_0 such that:

$$\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{n-k-1}\right)^{r_0}<\varepsilon \tag{13}$$

i.e.

$$r_0=\frac{\ln\varepsilon}{\ln\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{n-k-1}\right)} \tag{14}$$

Now, the approximated density function of T_k is:

$f_{k,n}(\lambda,\mu,t)\approx$

$$\sum_{r=0}^{r_0}\left(\frac{\mu}{\lambda+\mu}\right)^{r}\sum_{m\in A_r}\left(\frac{\lambda}{\lambda+\mu}\right)^{n-k-1+\sum_{n=2}^{n-1}(i-1)m_i}\frac{r!}{\prod_{i=1}^{n-k-1}m_i!}h(\lambda,r+1,\lambda+\mu,n-k-1+\sum_{i=1}^{n-k-1}im_i,t) \tag{15}$$

Similarly for the total failure time we get:

$f_n(\lambda,\mu,t)\approx$ (16)

$$\sum_{r=0}^{r_0}\left(\frac{\mu}{\lambda+\mu}\right)^{r}\sum_{m\in A_r}\left(\frac{\lambda}{\lambda+\mu}\right)^{n-1+\sum_{n=2}^{n-1}(i-1)m_i}\frac{r!}{\prod_{i=1}^{n-1}m_i!}h(\lambda,r+1,\lambda+\mu,n-1+\sum_{i=1}^{n-1}im_i,t)$$

4 Systems with several components

In this section is consider a system with M components (M>1), with equal transition densities, and the number of possible states of the j-th component is n_j+1. We suppose that components are mutually independent. The system can be repaired in such a way that after any repair all of the units are return back to their initial state, in fact after any repair, the system goes to its perfect state. It is also possible to repair a component that is in a state of total failure. We also suppose that if K of the components are in level 0 than the intensity of one level failure of the remaining components increases $\dfrac{M}{M-K}$ times. Consequently the sum of intensities of one level failure of all components is Mλ. Let the random variable T be the transition time from state (x_1, x_2, ..., x_i, ..., x_M) to state (x_1, x_2, ..., x_i-1, ..., x_M) of the system. Than:

$$T=min\{T^1, T^2, ..., T^M\} \qquad (17)$$

where T^j is the one level transition time of the j-th component. Since T^j for j=1, ..., M are i.i.d. random variables with exponential distribution with parameter $\dfrac{M}{M-K}\lambda$, T has an exponential distribution with parameter Mλ.

If we have a series system, the system total failure time is a minimum of the total failure times of its components. In that case reliability of the system is:

$$R(t)=\prod_{i=1}^{M} R_{n_i}(t) \qquad (18)$$

For a parallel system, system fails when all components are at state 0. The time for first entry of the system in this state, is the transition time needed for system to fall down from the state in which sum of all levels of the components is $N=n_1+n_2+...+n_M$ to state in which sum of all levels of components is 0. Reliability function for this type of system is:

$$R(t)=\int_t^{\infty} f_N(M\lambda,\mu,t)dt \qquad (19)$$

5 Estimation of the parameters

In this section we consider the problem for estimation the unknown parameters λ and μ in the density and reliability function from a given data that represent failure of the system. First note that we can have different kind of data, in fact we can know exactly with which level works the system in any time, or only when it is in the state of total failure, i.e. level 0. Also we can have the information how many times the system is repaired before its total failure, and from what level it is repaired. In this section we will give the methods for estimating unknown parameters in some of that different situation. We illustrate the procedure for a system with one component, because we see that the all system can be regarded as a one-component system.

First look at the case when the system is under control in any time. This means that we know the exact time for staying in any state and the following state of the system. The data consists of 4 parameters (i, $time_i$, $state1_i$, $state2_i$), were $time_i$ is the time of staying in $state1_i$ and $state2_i$ is the following state in which system or component goes after exiting $state1_i$. We know that the time for transition from state n to state n-1 has exponential distribution with parameterλ, and all other transitions have exponential distribution with parameter $\gamma=\lambda+\mu$. Then the ML estimator of the parameter λ from the data $t_i^{n,n-1}$ of times for transition of state n to n-1 is:

$$\hat{\lambda}_1 = \frac{a_{n,n-1}}{\sum_{i=1}^{a_{n,n-1}} t_i^{n,n-1}} \qquad (20)$$

Were $a_{n,n-1}$ is the number of all transitions from state n to state n-1. From the other hand if a is the number of all transitions and $t_i^{j,k}$ is time for i-th transition which is from state j to state k we have:

$$\hat{\gamma} = \frac{a - a_{n,n-1}}{\sum_{(j,k)\neq(n,n-1)} t_i^{j,k}} \qquad (21)$$

If we have $a_{n,n-1}$ transitions from state n to state n-1, than we have $a_{n,n-1}$-1 repairs. From here we can conclude that the probability for repair is equal to

$$\frac{a_{n,n-1} - 1}{a} \qquad (22)$$

and probability to failure is

$$\frac{a - a_{n,n-1} + 1}{a} \qquad (23)$$

From the other hand the repairs and failure probability is equal to $\frac{\mu}{\lambda + \mu}$ and $\frac{\lambda}{\lambda + \mu}$, and we estimate:

$$\hat{\mu}_2 = \frac{(a - a_{n,n-1})(a_{n,n-1} - 1)}{a \sum_{(j,k)\neq(n,n-1)} t_i^{j,k}} \text{ and } \hat{\lambda}_2 = \frac{(a - a_{n,n-1})(a - a_{n,n-1} + 1)}{a \sum_{(j,k)\neq(n,n-1)} t_i^{j,k}} \qquad (24)$$

Using (20) we make corrections in estimating parameters λ and μ as:

$$\hat{\lambda} = \frac{(a - a_{n,n-1})\hat{\lambda}_2 + a_{n,n-1}\hat{\lambda}_1}{a} \text{ and } \hat{\mu} = \hat{\gamma} - \hat{\lambda} \qquad (25)$$

Next case we regarded is when we only have data for total failure. Let T=T_0, then we have:

$$E(T) = \frac{1 - p^n}{\lambda p^{n-1}(1-p)} = \frac{1}{m} \sum_j t_j \qquad (26)$$

and

$$E(T^2)= \frac{1}{m}\sum_j t_j^2$$

$$= \frac{p^2}{\lambda^2}\left(\frac{2p^4-2p^3+2}{(1-p)^2p^{2n}} - \frac{n-2-(n-3)p}{1-p} - \frac{2n+2-(2n-5)p+(2n-4)p^2}{(1-p)p^n}\right) \qquad (27)$$

where $p=\frac{\lambda}{\lambda+\mu}$ and m is the data number. From here we can estimate unknown parameters but solving that system of equation is not easy at all. For that reason we give the ways for estimating parameters when we know how many repairs we have and from where it is.

Let we have $b_{i,j}$ repairs from i-th level in j-th total failure. Set $b_i=\frac{1}{m}\sum_j b_{i,j}$. Then the mean of total partial failures is $n+\sum_{i=1}^{n-1}(n-i)b_i$ and the mean of total partial repairs is $\sum_{i=1}^{n-1} b_i$. For the probability of failure we have:

$$\hat{p}=\frac{\hat{\lambda}}{\hat{\lambda}+\hat{\mu}}=\frac{n+\sum_{i=1}^{n-1}(n-i)b_i-\sum_{i=1}^{n-1}b_i-1}{n+\sum_{i=1}^{n-1}(n-i)b_i-1} \qquad (28)$$

and

$$\hat{\lambda}=\frac{m(1-\hat{p}^n)}{\hat{p}^{n-1}(1-\hat{p})\sum_j t_j} \qquad (29)$$

At the last we consider the case when we know only how many repairs we have. Let the mean of repairs is equal to *N*. The probability to have one repair is equal to $1-p^{n-1}$, and the probability to have repair from i-th level if we already have repair is:

$$\frac{p^{n-i-1}(1-p)}{1-p^{n-1}} \qquad (30)$$

From here the expected number of repairs from i-th level is:

$$b_i=\frac{p^{n-i-1}(1-p)}{1-p^{n-1}}N \qquad (31)$$

When we took this in (28) we obtain:

$$\hat{p} = \frac{(n-N-1)(1-\hat{p})(1-\hat{p}^{n-1}) + N(1-n\hat{p}^{n-1}+(n-1)\hat{p}^{n})}{(n-1)(1-\hat{p})(1-\hat{p}^{n-1}) + N(1-n\hat{p}^{n-1}+(n-1)\hat{p}^{n})} \tag{32}$$

From this and equation (29) we can obtain the unknown intensities.

After estimation of intensities, we are going to estimate the density function of the process we have.

Example1: We have simulated data for system with set of levels E={0,1,2,3,4} with repair intensity 1 and failure intensity 3.5. The estimated intensities, in the case when we have system which is under control in any time are $\hat{\lambda}$=3.5144 and $\hat{\mu}$=1.0992. Next figures give the histogram and estimate approximate density functions.

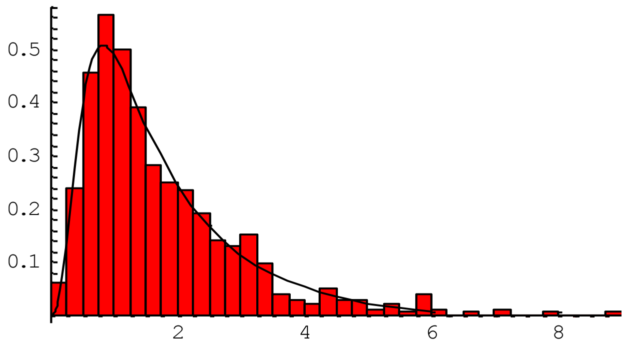

Fig1: For $\hat{\lambda}$=3.5144, $\hat{\mu}$=1.0992 and first entry in level 0

Example2: For simulated data for system with set of levels E={0,1,2,3} with repair intensity 2 and failure intensity 5. The estimated intensities, in the case when we have system which is under control in any time are $\hat{\lambda}_1$=5.02 and $\hat{\mu}_1$=1.98. Next figures give the histogram and estimate approximate density functions.

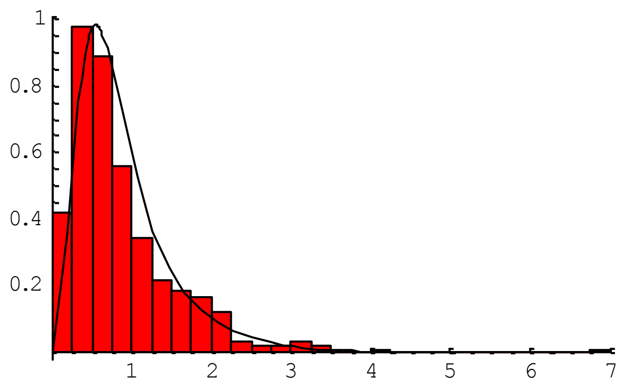

Fig 2: For $\hat{\lambda}_1$=5.02, $\hat{\mu}_1$=1.98

The estimated intensities in the case when we know number of repairs are $\hat{\lambda}_2$=5.03 and $\hat{\mu}_2$=1.97. The histogram and estimated density function for this case is on the following picture:

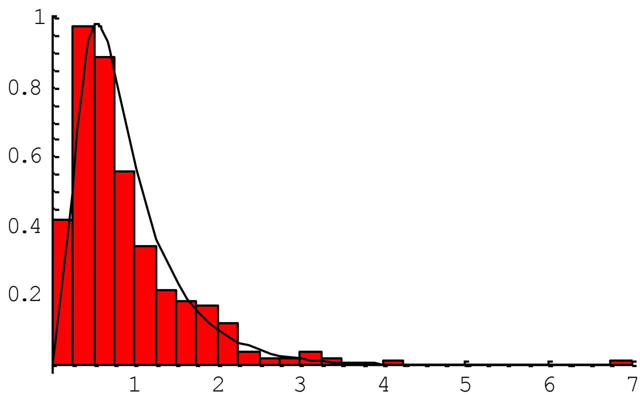

Fig 3: For $\hat{\lambda}_2$=5.03, $\hat{\mu}_2$=1.97

6 Conclusion

In this paper we regard a model of multi-state repairable system. We suppose that the system work in that way so its quality decrease step by step with same intensity of all levels and it can be repaired only to the perfect state, also with same intensity for all levels. We are concentrate to the reliability of that system and look for mathematical tools for calculating reliability function for a given system from given data, when we assume that the system is of the kind we propose.

7 References

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