Multi-state Systems with Graduate Failure and Equal Transition Intensities

Marija Mihova and Zaneta Popeska

Abstract. We consider unrecoverable homogeneous multi-state systems with graduate failures, where each component can work at $M + 1$ linearly ordered levels of performance. The underlying process of failure for each component is a homogeneous Markov process such that the level of performance of one component can change only for one level lower than the observed one, and the failures are independent for different components. We derive the probability distribution of the random vector $\mathbf{X}$, representing the state of the system at the moment of failure and use it for testing the hypothesis of equal transition intensities. Under the assumption that these intensities are equal, we derive the method of moments estimators for probabilities of failure in a given state vector and the intensity of failure. At the end we calculate the reliability function for such systems.

Mathematics Subject Classification (2000). Primary 62N05; Secondary 62N02.

Keywords. Multi-state system, minimal path set, minimal cut set, reliability function, failure intensities.

1. Introduction

The statistical approach towards reliability usually is based on determination of probability model that can give the best description of the work of the whole system without consideration of its structure [2]. Nevertheless, in the literature one can find a lot of papers in which, in analyzing the system reliability, it is taken in consideration the components of the system and their interaction. But, usually they are binary systems or systems consisting of binary components [1], [3], [7], while the multi-state systems are less analyzed. Analysis of multi-state systems is mainly based on expressing the reliability of the system using the reliability for certain level of the components without the consideration of the time as a dimension in the working process of the system [6] and [7]. In these papers only the influence of the structure of the system on its reliability is considered. Some

of the rare attempts to analyze the reliability of the multi-state systems based on the working process of the system are the papers [4] and [5], where the target are monotone homogeneous recoverable systems. If the work of the multi-state system is considered as a continuous process through the time, then the calculation of the reliability function becomes very complex even in the simplest systems. Determination of some relationships between the unknown parameters can be used to simplify the calculation of the reliability function.

2. Basic definitions

Consider a multi-state system with n components, such that each component can be in one of the $M + 1$ levels, where M is the level of a perfect state of the component and 0 is the level of its total failure. The coordinate x_i of the state vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ represents the state of the i-th component of the system, for $0 \leq x_i \leq M$ and $1 \leq i \leq n$. Let $S = \{\mathbf{x} | 0 \leq x_i \leq M, \ 1 \leq i \leq n\}$ be the state set of the system. The state vector for which all components are in a perfect state M is denoted by $\mathbf{M}$. All vectors in S that define a working state of the system are called **path** vectors and the vectors in S for which the system does not work are called **cut** vectors. Let $\overline{\mathcal{P}}$ be the set of all path vectors and $\overline{\mathcal{C}}$ the set of all cut vectors.

Define an ordering of the set S by $\mathbf{x} \leq \mathbf{y}$ iff $\forall i \ 1 \leq i \leq n, \ x_i \leq y_i$. We assume that if $\mathbf{x}$ is a path vector, then all vectors greater than $\mathbf{x}$ are path vectors, and any vector smaller than some cut vector is also a cut vector. This means that if the system is in a working state $\mathbf{x}$, it will work in all states greater than $\mathbf{x}$ and if it does not work in some state, then it does not work in all states smaller than this one. This kind of systems are known as monotone multi-state systems (MMS). In this paper we regard MMS systems that consist of components with graduate failure, i.e. the process of failure is moving one "step" at a time, by changing one component by one level down at each move. Moreover we assume that the components can not be repaired during the work of the system, i.e. we consider unrecoverable systems. We give the following definitions:

Definition 2.1. A vector $\mathbf{x}$ is a **minimal path** vector iff it is a path vector and all vectors smaller than $\mathbf{x}$ are cut vectors. A vector $\mathbf{x}$ is a **minimal cut** vector iff it is a cut vector and all vectors greater than $\mathbf{x}$ are path vectors.

The set of all minimal path vectors, $\mathcal{P}$, is called a **minimal path set** and the set of all minimal cut vectors, $\mathcal{C}$, is called a **minimal cut set**.

Definition 2.2. A vector $\mathbf{y}$ is a **fatal vector** iff it is a cut vector and there is a path vector $\mathbf{x}$ such that $\mathbf{x} - \mathbf{y} = \mathbf{e}_i$, where $\mathbf{e}_i$ is the unit n-vector with 1 as i-th component, and 0 elsewhere.

The set of all fatal vectors is called a **fatal set**, denoted by $\mathcal{C}_f$.

Example. For 3-component system with $\mathbf{M} = (2,2,2)$, $\mathcal{P} = \{(1,1,1)\}$ and $\mathcal{C} = \{(0,2,2),(2,0,2),(2,2,0)\}$, the set $\mathcal{C}_f = \{(0,2,2),(2,0,2),(2,2,0),(2,1,0),(1,2,0),$
$(1,1,0),(0,1,2),(0,2,1),(0,1,1),(1,0,2),(2,0,1),(1,0,1)\}$.

The sequence: $\mathbf{x} = \mathbf{a}^0, \mathbf{a}^1, \dots, \mathbf{a}^k = \mathbf{y}$ where $\mathbf{a}^j = \mathbf{a}^{j-1} - \mathbf{e}_i$ for some $i = \overline{1,n}$ is called a **path** from the state $\mathbf{x}$ to the state $\mathbf{y}$, denoted by $\gamma_{\mathbf{a}^0, \mathbf{a}^1, \dots, \mathbf{a}^k}$. Let $\Gamma(\mathbf{x})$ be the set of all paths from $\mathbf{M}$ to $\mathbf{x}$ and $\Gamma(\mathbf{x}, \mathbf{y})$, the set of all paths from $\mathbf{x}$ to $\mathbf{y}$.

For each state vector $\mathbf{x}$ we define a vector $\widetilde{\mathbf{x}}$ and a number $n_{\mathbf{x}}$ by:

$$\widetilde{\mathbf{x}} = \mathbf{M} - \mathbf{x}, \quad n_{\mathbf{x}} = \sum_{i=1}^{n} \widetilde{x}_i. \tag{2.1}$$

The number $n_{\mathbf{x}}$ represents the **length of the path** from $\mathbf{M}$ to $\mathbf{x}$, i.e. the number of visited states from $\mathbf{M}$ to $\mathbf{x}$. Each path is uniquely determined by a vector, via the mapping $\varphi_{\mathbf{x}} : \Gamma(\mathbf{x}) \to \{1,2,\dots,n\}^{n_{\mathbf{x}}}$ defined by

$$\varphi_{\mathbf{x}}(\gamma_{\mathbf{M}=\mathbf{a}^0,\mathbf{a}^1,\dots,\mathbf{a}^{n_{\mathbf{x}}}=\mathbf{x}}) = \mathbf{v}, \text{ where } v_j = i \text{ iff } \mathbf{a}^{j-1} - \mathbf{a}^j = \mathbf{e}_i. \tag{2.2}$$

Note that the j-th coordinate of the vector $\mathbf{v}$ represents the component that degrades in the j-th step. The set $\varphi(\Gamma(\mathbf{x}))$ is the set of all vectors in $\mathbb{N}_n^{n_{\mathbf{x}}}$ with the property, $\widetilde{x}_i$ of its coordinates are equal to i. We denote the set $\varphi(\Gamma(\mathbf{x}))$ by $V_{\mathbf{x}}$ and $\varphi(\gamma) = \mathbf{v}_{\gamma}$.

Example. Let $\mathbf{M} = (3,3,3)$, $\mathbf{x} = (3,2,1)$ and the path γ from $\mathbf{M}$ to $\mathbf{x}$ is γ : $(3,3,3),(3,2,3),(3,2,2),(3,2,1)$, then $\varphi(\gamma) = (2,3,3)$.

The system starts at the perfect state and at the moment of failure it is in some state from the set $\mathcal{C}_f$. We suppose that we can observe the time of failure of the system and its state vector at that moment, but the path to this state is unknown.

Let $p_{\mathbf{x},\mathbf{y}}$ be the transition probability from a state $\mathbf{x}$ to a state $\mathbf{y}$ and $\lambda_{i,j}, i = \overline{1,n}, j = \overline{1,M_i}$, be the one step transition intensity from state j to state $j-1$ of the i-th component. Then,

$$p_{\mathbf{x},\mathbf{x}-\mathbf{e}_i} = \frac{\lambda_{i,x_i}}{\displaystyle\sum_{1\leq j\leq n,\ x_j\neq 0} \lambda_{j,x_j}}. \tag{2.3}$$

By $p_{\mathbf{x}} = p_{\mathbf{M},\mathbf{x}}$ we will note the probability of all possible paths to state $\mathbf{x}$. Note that if $\mathbf{x}$ is a cut vector which is not a fatal vector, then $p_{\mathbf{x}} = 0$. For all other state vectors $\mathbf{x}$, $p_{\mathbf{x}} > 0$.

Let $\mathbf{X}$ be the random vector representing the state of the system at the moment of failure. As mentioned earlier, when the system fails, it is found in some of the elements of the set $\mathcal{C}_f$. The probability that the system is in a state $\mathbf{c} \in \mathcal{C}_f$ at the moment of failure, $P(\mathbf{X} = \mathbf{c})$, is equal to $p_{\mathbf{c}}$.

Definition 2.3. The probability distribution of the random variable $\mathbf{X}$ on the set $\mathcal{C}_f$ is called **failure distribution**.

Let $\mathbf{c} \in \mathcal{C}_f$ and $\gamma = \gamma_{\mathbf{M}, \mathbf{a}^1, \ldots, \mathbf{a}^{n_\mathbf{c}} = \mathbf{c}}$ be a path from $\mathbf{M}$ to $\mathbf{c}$. Then, the probability that the system will fail throw the path γ, $\widetilde{p}_\gamma$ is:

$$\widetilde{p}_\gamma = \prod_{j=1}^{n_\mathbf{c}} p_{\mathbf{a}^j, \mathbf{a}^{j+1}}. \qquad (2.4)$$

Now, for $p_\mathbf{c}$ we have:

$$p_\mathbf{c} = \sum_{\gamma \in \Gamma(\mathbf{c})} \widetilde{p}_\gamma. \qquad (2.5)$$

The formula derived can be used in order to test the hypothesis of equal transition intensities.

3. Testing hypothesis of equal transition intensities

Suppose that we have N independent systems for which we look at their state at the moment of failure, which is some state from the set $\mathcal{C}_f$. By $f_\mathbf{c}$ we denote the observed frequency of the systems that fail in a given state $\mathbf{c}$. If $|\mathcal{C}_f| = m$, then

$$\sum_{\mathbf{c} \in \mathcal{C}_f} \frac{(N p_\mathbf{c} - f_\mathbf{c})^2}{N p_\mathbf{c}} \sim \chi^2_{m-1}. \qquad (3.1)$$

Therefore, using χ^2 test we can test the hypothesis that the intensities of the system are equal. In order to use this test, we have to derive the failure distribution of the system in which all of the one step transition intensities of the components are equal, i.e. $\forall i = \overline{1, n}$ and $\forall j = \overline{1, M}$, $\lambda_{i,j} = \lambda$, for $\lambda \in \mathbb{R}^+$. Let $k_\mathbf{x}$ be the number of zero coordinates of the vector state $\mathbf{x}$, then from (2.3) we have:

$$p_{\mathbf{x}, \mathbf{x} - \mathbf{e}_i} = \frac{\lambda}{(n - k_\mathbf{x}) \lambda} = \frac{1}{(n - k_\mathbf{x})}. \qquad (3.2)$$

Note that in this type of systems the one step transition probabilies do not depend on the intensities.

Suppose that the set $\mathcal{C}_f$ is known and $\mathbf{c} \in \mathcal{C}_f$. For $p_\mathbf{c}$ we have:

$$p_\mathbf{c} = \sum_{\mathbf{x} \in \mathcal{P}, \mathbf{x} = \mathbf{c} + \mathbf{e}_i} p_\mathbf{x} p_{\mathbf{x}, \mathbf{c}}. \qquad (3.3)$$

Next we will find the probability $p_\mathbf{x}$ for all $\mathbf{x} \in S$. Since the one level transition probabilities do not depend on λ, $p_\mathbf{x}$ does not depend on λ also. We consider two types of state vectors: the first one, when all coordinates of the vector are different from zero and the second one, when some of them are equal to zero.

When all coordinates of the vector $\mathbf{x}$ are different from zero, all the paths to this vector have equal probabilities, otherwise, from (3.2) it follows that the probabilities are not equal. We will illustrate this by the following example.

Example. Let $\mathbf{M} = (3, 3, 3)$ and $\mathbf{x} = (2, 2, 0)$. Look at the paths: $\gamma_1 : (3, 3, 3)$, $(3, 3, 2)$, $(3, 2, 2)$, $(3, 2, 1)$, $(3, 2, 0)$, $(2, 2, 0)$ and γ_2: $(3, 3, 3)$, $(3, 3, 2)$, $(3, 2, 2)$, $(3, 2, 1)$, $(2, 2, 1)$, $(2, 2, 0)$. The probability $\widetilde{p}_{\gamma_1} = \left(\frac{1}{3}\right)^3 \frac{1}{2}$ and the probability $\widetilde{p}_{\gamma_2} = \left(\frac{1}{3}\right)^4$.

Theorem 3.1. *If, in a given MMS system, all coordinates of the vector $\mathbf{x}$ are different from zero, then*

$$p_{\mathbf{x}} = \frac{n_{\mathbf{x}}!}{n^{n_{\mathbf{x}}} \prod_{i=1}^n \widetilde{x}_i!}. \tag{3.4}$$

When the vector $\mathbf{x}$ has zero coordinates then

$$p_{\mathbf{x}} = A(\mathbf{x}) \sum_{j_1=a(1)}^{b(1)} \sum_{j_2=a(2)}^{b(2)} \ldots \sum_{j_{k_{\mathbf{x}}}=a(k_{\mathbf{x}})}^{b(k_{\mathbf{x}})} \prod_{s=1}^{k_{\mathbf{x}}} B(s, j_s). \tag{3.5}$$

where

$$
\begin{aligned}
A(\mathbf{x}) &= \frac{k_{\mathbf{x}}!}{(n - k_{\mathbf{x}})^{n_{\mathbf{x}}}} \frac{(n_{\mathbf{x}} - k_{\mathbf{x}}M)!}{\prod_{x_i \neq 0} \widetilde{x}_i!}; \\
B(s, j_s) &= \binom{j_s - (s-1)M - 1}{M - 1} \left(\frac{n - s}{n - s + 1}\right)^{j_s}; \\
a(1) &= M; \\
a(s) &= \max(sM, j_{s-1} + 1), \ s > 1; \\
b(s) &= n_{\mathbf{x}} - (k_{\mathbf{x}} - s).
\end{aligned}
\tag{3.6}
$$

Proof. When all the coordinates of the vector $\mathbf{x}$ are different from zero, then all the paths have probability $\left(\frac{1}{n}\right)^{n_{\mathbf{x}}}$. So, all we need is to find the number of paths, i.e $|V_{\mathbf{x}}|$. This number is equal to $\frac{(\sum_{i=1}^n \widetilde{x}_i)!}{\prod_{i=1}^n \widetilde{x}_i!}$. Consequently, we obtain (3.4):

$$p_{\mathbf{x}} = \frac{(\sum_{i=1}^n \widetilde{x}_i)!}{n^{n_{\mathbf{x}}} \prod_{i=1}^n \widetilde{x}_i!} = \frac{n_{\mathbf{x}}!}{n^{n_{\mathbf{x}}} \prod_{i=1}^n \widetilde{x}_i!}.$$

Now consider the case when the vector $\mathbf{x}$ has coordinates equal to 0. The probability of a path depends on the steps in which the components enter at the level of total failure. Let γ be a path from M to $\mathbf{x}$. If the i-th coordinate of the vector $\mathbf{x}$ is equal to 0, then exactly M coordinates of the vector $\mathbf{v}_\gamma$ have value i. So, if $\mathbf{x}$ has $k_{\mathbf{x}}$ coordinates equal to 0, then there are exactly $k_{\mathbf{x}}$ groups of M equal value coordinates in the vector $\mathbf{v}_\gamma$. In fact if the coordinates which are equal to 0 in the vector $\mathbf{x}$ are: $i_1, i_2, \ldots, i_{k_{\mathbf{x}}}$, then the numbers $i_1, i_2, \ldots, i_{k_{\mathbf{x}}}$ are found M times in the vector $\mathbf{v}_\gamma$. We are interested in the M-th appearance of each one.

Let the M-th appearance of the number i_s, $s = \overline{1, k_{\mathbf{x}}}$ in the vector $\mathbf{v}_\gamma$ be at the j_s-th place i.e. as the j_s-th coordinate. We can reorder the numbers $i_1, i_2, \ldots, i_{k_{\mathbf{x}}}$ such that $j_1 \leq j_2 \leq \ldots \leq j_{k_{\mathbf{x}}}$. There are $k_{\mathbf{x}}!$ ways of ordering of these numbers. The number of ways for obtaining the vector $\mathbf{v}$ such that the M-th appearance of the value i_s is at the j_s-th place is equal to

$$\begin{pmatrix} j_1 - 1 \\ M - 1 \end{pmatrix} \begin{pmatrix} j_2 - M - 1 \\ M - 1 \end{pmatrix} \cdots \begin{pmatrix} j_{k_{\mathbf{x}}} - (k_{\mathbf{x}} - 1)M - 1 \\ M - 1 \end{pmatrix} \frac{(n_{\mathbf{x}} - k_{\mathbf{x}}M)!}{\prod_{x_i \neq 0} \widetilde{x}_i!}$$

$$= \frac{(n_{\mathbf{x}} - k_{\mathbf{x}}M)!}{\prod_{x_i \neq 0} \widetilde{x}_i!} \prod_{s=1}^{k_{\mathbf{x}}} \begin{pmatrix} j_s - (s - 1)M - 1 \\ M - 1 \end{pmatrix}.$$

(3.7)

There of the following inequalities must be satisfied:

$$\begin{aligned} M \leq \quad & j_1 \quad \leq n_{\mathbf{c}} - (k_{\mathbf{x}} - 1) \\ \max(sM, j_{s-1} + 1) \leq \quad & j_s \quad \leq n_{\mathbf{x}} - (k_{\mathbf{x}} - s), \ s > 1 \end{aligned}$$

(3.8)

The probability $\widetilde{p}_\gamma$ is:

$$\begin{aligned} \widetilde{p}_\gamma \ &= \left(\frac{1}{n}\right)^{j_1} \left(\frac{1}{n-1}\right)^{j_2 - j_1} \cdots \left(\frac{1}{n - (k_{\mathbf{x}} - 1)}\right)^{j_{k_{\mathbf{x}}-1} - j_{k_{\mathbf{x}}-2}} \left(\frac{1}{n - k_{\mathbf{x}}}\right)^{n_{\mathbf{x}} - j_{k_{\mathbf{x}}}} \\ &= \left(\frac{1}{n - k_{\mathbf{x}}}\right)^{n_{\mathbf{x}}} \prod_{s=1}^{k_{\mathbf{x}}} \left(\frac{n-s}{n-s+1}\right)^{j_s} \end{aligned}$$

(3.9)

Using (3.7), (3.8), (3.9) and the fact that the numbers $i_1, i_2, \ldots, i_{k_{\mathbf{x}}}$ can be ordered in $k_{\mathbf{x}}!$ number of ways, we have:

$$p_{\mathbf{x}} = A(\mathbf{x}) \sum_{j_1 = a(1)}^{b(1)} \sum_{j_2 = a(2)}^{b(2)} \cdots \sum_{j_{k_{\mathbf{x}}} = a(k_{\mathbf{x}})}^{b(k_{\mathbf{x}})} \prod_{s=1}^{k_{\mathbf{x}}} B(s, j_s),$$

where $A(\mathbf{x})$, $B(s, j_s)$, $a(1)$, $a(s)$ and $b(s)$ are given by (3.6). $\qquad\square$

Equation (3.5) can be written as:

$$p_{\mathbf{x}} = A(\mathbf{x}) \sum_{j_1 = a(1)}^{b(1)} B(1, j_1) \sum_{j_2 = a(2)}^{b(2)} B(2, j_2) \ldots \sum_{j_{k_{\mathbf{x}}} = a(k_{\mathbf{x}})}^{b(k_{\mathbf{x}})} B(k_{\mathbf{x}}, j_{k_{\mathbf{x}}}).$$

This representation is useful for obtaining the algorithm for calculating $p_{\mathbf{c}}$, for a given fatal vector $\mathbf{c}$. The algorithm is given by:

1. Put the number of components n and the maximum working level M;
2. Define a function $\mathrm{comb}(\mathbf{x}) = \prod_{x_i \neq 0}(M - x_i)!$;
3. Define a function $n_{\mathbf{x}}$, given by (2.1);
4. Define a function $k_{\mathbf{x}}$, number of zero coordinates of the state vector $\mathbf{x}$;

5. Define a function $A(\mathbf{x}) = \dfrac{k_{\mathbf{x}}!}{(n - k_{\mathbf{x}})^{n_{\mathbf{x}}}} \dfrac{(n_{\mathbf{x}} - k_{\mathbf{x}}M)!}{\mathrm{comb}(\mathbf{x})}$, given by (3.6);

6. Define $B(s, j) = \begin{pmatrix} j - (s-1)M - 1 \\ M - 1 \end{pmatrix} \left(\dfrac{n-s}{n-s+1}\right)^j$, given by (3.6);

7. Define $L(\mathbf{x}, s, j) = \begin{cases} B(s, j), & s = k_{\mathbf{x}}, \\ B(s, j) \displaystyle\sum_{u=\mathrm{Max}((s+1)M, j+1)}^{n_{\mathbf{x}} - k_{\mathbf{x}} + s + 1} L(\mathbf{x}, s+1, u), & s < k_{\mathbf{x}}; \end{cases}$

8. $p_{\mathbf{x}} = \begin{cases} A(\mathbf{x}), & k_{\mathbf{x}} = 0, \\ A(\mathbf{x}) \displaystyle\sum_{u=M}^{n_{\mathbf{x}} - k_{\mathbf{x}} + 1} L(\mathbf{x}, 1, u), & k_{\mathbf{x}} \neq 0; \end{cases}$

9. Define the set $D_{\mathbf{c}}$ of all vectors $\mathbf{c} + \mathbf{e}_i \in S \setminus \mathcal{C}_f$;

10. $p_{\mathbf{c}} = \displaystyle\sum_{\mathbf{x} \in \mathcal{P}, \mathbf{x} = \mathbf{c} + \mathbf{e}_i} \frac{p_{\mathbf{x}}}{n - k_{\mathbf{x}}}.$

Next some experimental results are presented. We performed the test on two types of simulated systems: with equal intensities and with different intensities. The results are given below:

Example. Nine groups of 1000 3-component system with $\mathbf{M} = (3, 3, 3)$, $\mathcal{C} = \{(3, 0, 1), (1, 2, 1), (0, 1, 3), (1, 3, 0), (3, 1, 0), (0, 3, 1), (1, 0, 3)\}$ and all failure intensities equal to 1 are simulated. The fatal set of this system is $\mathcal{C}_f = \{(0,1,2), (0,1,3), (0,2,1), (0,3,1), (1,0,2), (1,0,3), (1,1,1), (1,2,0), (1,2,1), (1,3,0), (2,0,1), (2,1,0), (3,0,1), (3,1,0)\}$. In Table 1 the values of the χ^2-statistics, (3.1), and the significant levels of acceptance, obtained from those experiments are given. It is obvious that the hypotheses that all parameters are equal has been accepted in all cases.

exp	1	2	3	4	5	6	7	8	9
χ^2_{13}	10.38	23.17	13.71	8.01	11.17	14.67	13.28	21.87	3.3
p	0.66	0.04	0.4	0.84	0.6	0.33	0.43	0.06	0.996

Table 1

Example. Two types of systems with $\mathbf{M}$ and $\mathcal{C}$ as in the previous example are simulated. In the first type of system, the failure intensities $q_{1,3} = 0.5$ and $q_{1;2} = 1.5$ and the other are equal to 1. In the second, $q_{1;3} = q_{2;3} = q_{3;3} = 2$, and the other intensities are equal to 1. There are simulated 5 groups of 1000 systems of both type. Table 2 gives the value of the χ^2-statistics given by (3.1). The significant level of acceptance is always close to 0 and the hypothesis is not accepted.

exp	1	2	3	4	5
First system: χ^2_{13}	208.19	188.32	176.21	235.458	254.78
Second system: χ^2_{13}	97.7	108.19	73.03	81.99	63.84

Table 2

4. Failure intensity estimator

Suppose that we have indications that all of the intensities of the system are equal, i.e $q_{i,j} = q$, $\forall i,j$ (by using the test we explained earlier or in some other manner). Now we estimate this parameter. In this section we will give an estimator of the intensity of failure q, using the method of moments.

Theorem 4.1. *Let $t_1, t_2, \ldots, t_N$ be a random sample of failure times of the MMS system in which all transition intensities are equal. Then the estimator by method of moments of intensity q is given by:*

$$\widehat{q} = \frac{N}{\sum_{i=1}^{N} t_i} \sum_{\mathbf{c} \in \Gamma} \sum_{\mathbf{x} \in S \setminus C_f, \mathbf{x} = \mathbf{c} + \mathbf{e}_i} \frac{1}{n - k_{\mathbf{x}}} \left(D_{\mathbf{x}} + \frac{p_{\mathbf{x}}}{n - k_{\mathbf{x}}} \right), \qquad (4.1)$$

where

$$D_{\mathbf{x}} = \left(\frac{n_{\mathbf{x}} p_{\mathbf{x}}}{n - k_{\mathbf{x}}} - A(\mathbf{x}) \sum_{j_1 = a(1)}^{b(1)} \sum_{j_2 = a(2)}^{b(2)} \cdots \sum_{j_{k_{\mathbf{x}}} = a(k_{\mathbf{x}})}^{b(k_{\mathbf{x}})} \sum_{s=1}^{k_{\mathbf{x}}} \frac{j_s \prod_{s=1}^{k_{\mathbf{x}}} B(s, j_s)}{(n - s)(n - s + 1)} \right)$$

and $A(\mathbf{x})$, $B(s, j_s)$, $a(s)$ and $b(s)$ are given by (3.6) and $p_{\mathbf{x}}$ is given by (3.5).

Proof. First we define five random variables, T the time to failure of the system, G the path to failure and T_γ the time of failure through the path γ, $U_{\mathbf{x},\mathbf{y}}$, the transition time from the state $\mathbf{x}$ to the state $\mathbf{y}$, and $U_{\mathbf{x}} = U_{\mathbf{M},\mathbf{x}}$. Let $F(t) = P(T < t)$, $f(t) = F'(t)$, $F_\gamma(t) = P(T_\gamma < t) = P(T < t | G = \gamma)$ and $F_\gamma(t) = F'_\gamma(t)$. Then:

$$
\begin{aligned}
F(t) &= P(T < t) = P \left(\bigcup_{\gamma \in \Gamma} (T < t, G = \gamma) \right) = \sum_{\gamma \in \Gamma} P(T < t, G = \gamma) \\
&= \sum_{\gamma \in \Gamma} P(T < t | G = \gamma) \widetilde{p}_\gamma = \sum_{\gamma \in \Gamma} P(T_\gamma < t) \widetilde{p}_\gamma = \sum_{\gamma \in \Gamma} F'_\gamma(t) \widetilde{p}_\gamma.
\end{aligned}
\qquad (4.2)
$$

and

$$f(t) = F'(t) = \sum_{\gamma \in \Gamma} F'_\gamma(t) \widetilde{p}_\gamma = \sum_{\gamma \in \Gamma} f_\gamma(t) \widetilde{p}_\gamma. \qquad (4.3)$$

Using this, for the first moment we have

$$E(T) = \int_0^\infty t f(t) \mathrm{d}t = \int_0^\infty t \sum_{\gamma \in \Gamma} f_\gamma(t) \widetilde{p}_\gamma = \sum_{\gamma \in \Gamma} \int_0^\infty t f_\gamma(t) \widetilde{p}_\gamma = \sum_{\gamma \in \Gamma} E(T_\gamma) \widetilde{p}_\gamma. \qquad (4.4)$$

The last equation can be written as

$$E(T) = \sum_{\mathbf{c} \in C_f} \sum_{\gamma \in \Gamma(\mathbf{c})} E(T_\gamma) \widetilde{p}_\gamma = \sum_{\mathbf{c} \in C_f} E(U_{\mathbf{c}}). \qquad (4.5)$$

Suppose that $\gamma = \gamma_{\mathbf{a}^0 = \mathbf{M}, \mathbf{a}^1, \ldots, \mathbf{a}^{n_{\mathbf{c}}} = \mathbf{c}}$. Then

$$E(T_\gamma) = E \left(\sum_{i=1}^{n_{\mathbf{x}}} U_{\mathbf{a}^{i-1}, \mathbf{a}^i} \right) = \sum_{i=1}^{n_{\mathbf{x}}} E(U_{\mathbf{a}^{i-1}, \mathbf{a}^i}). \qquad (4.6)$$

Let $W_{\mathbf{x}}^i$ be the random variable that represents the transition time of the i-th component from the level x_i to the level x_{i-1}. $W_{\mathbf{x}}^i$ has an exponential distribution with parameter λ, for all i and all $\mathbf{x}$, for which $x_i \neq 0$. Therefore, since the components work independently and $U_{\mathbf{x},\mathbf{x}-\mathbf{e}_i} = \min_i W_{\mathbf{x}}^i$ we have that $U_{\mathbf{x},\mathbf{x}-\mathbf{e}_i}$ has an exponential distribution with parameter $(n - k_{\mathbf{x}})\lambda$. Then

$$
\begin{aligned}
E(U_{\mathbf{c}}) &= \sum_{\gamma \in \Gamma(\mathbf{c})} E(T_\gamma)\widetilde{p}_\gamma = \sum_{\mathbf{x} \in S \setminus C_f, \mathbf{x}=\mathbf{c}+\mathbf{e}_i} \sum_{\gamma \in \Gamma(\mathbf{x})} E(T_\gamma + U_{\mathbf{x},\mathbf{c}})\widetilde{p}_\gamma p_{\mathbf{x},\mathbf{c}} \\
&= \sum_{\mathbf{x} \in S \setminus C_f, \mathbf{x}=\mathbf{c}+\mathbf{e}_i} p_{\mathbf{x},\mathbf{c}} \left(\sum_{\gamma \in \Gamma(\mathbf{c})} E(T_\gamma)\widetilde{p}_\gamma + E(U_{\mathbf{x},\mathbf{c}}) \sum_{\gamma \in \Gamma(\mathbf{c})} \widetilde{p}_\gamma \right) \\
&= \sum_{\mathbf{x} \in S \setminus C_f, \mathbf{x}=\mathbf{c}+\mathbf{e}_i} \frac{1}{n - k_{\mathbf{x}}} \left(\sum_{\gamma \in \Gamma(\mathbf{c})} E(T_\gamma)\widetilde{p}_\gamma + \frac{1}{(n - k_{\mathbf{x}})\lambda} p_{\mathbf{x}} \right).
\end{aligned} \tag{4.7}
$$

When $\mathbf{x}$ has no zero coordinates:

$$
E(T_\gamma) = \frac{\displaystyle\sum_{i=1}^n \widetilde{\mathbf{x}}_i}{n\lambda} = \frac{1}{\lambda} \cdot \frac{n_{\mathbf{x}}}{n}. \tag{4.8}
$$

To obtain the expression for $E(T_\gamma)$ in the case when there are zero coordinates in $\mathbf{x}$, we use the vector $\mathbf{v}_\gamma$. If the coordinates which are equal to 0 in the vector $\mathbf{x}$ are: $i_1, i_2, \ldots, i_{k_{\mathbf{x}}}$, the numbers $i_1, i_2, \ldots, i_{k_{\mathbf{x}}}$ are found M times in the vector $\mathbf{v}_\gamma$. Suppose that the M-th appearance of the number i_s, $s = \overline{i, k_{\mathbf{x}}}$ in the vector $\mathbf{v}_\gamma$ is at the j_s-th position, then

$$
\begin{aligned}
E(T_\gamma) &= \frac{j_1}{n\lambda} + \frac{j_2 - j_1}{(n-1)\lambda} + \frac{j_3 - j_2}{(n-2)\lambda} + \ldots + \frac{j_{k_{\mathbf{x}}} - j_{k_{\mathbf{x}}-1}}{(n - k_{\mathbf{x}} + 1)\lambda} + \frac{n_{\mathbf{x}} - j_{k_{\mathbf{x}}}}{(n - k_{\mathbf{x}})\lambda} \\
&= \frac{1}{\lambda} \left(\frac{n_{\mathbf{x}}}{n - k_{\mathbf{x}}} - \sum_{i=1}^{k_{\mathbf{x}}} \frac{j_i}{(n - i)(n - i + 1)} \right).
\end{aligned} \tag{4.9}
$$

Now, for the state vector $\mathbf{x} \in \mathcal{P}$, we will find the sum

$$
E(U_{\mathbf{x}}) = \sum_{\gamma \in \Gamma(\mathbf{x})} E(T_\gamma)\widetilde{p}_\gamma. \tag{4.10}
$$

When there are no zeros in the vector $\mathbf{x}$, from (3.4) this sum is equal to:

$$
E(U_{\mathbf{x}}) = \frac{n_{\mathbf{x}}}{n\lambda} \frac{n_{\mathbf{x}}!}{n^{n_{\mathbf{x}}} \prod_{i=1}^n \widetilde{x}_i!} = \frac{1}{\lambda} \cdot \frac{n_{\mathbf{x}} n_{\mathbf{x}}!}{n^{n_{\mathbf{x}}+1} \prod_{i=1}^n \widetilde{x}_i!} \tag{4.11}
$$

When there are zeros in the vector $\mathbf{x}$, the value of $\widetilde{p}_{\gamma_{\mathbf{x}}}$ is given by the (3.9), and because we can choose the order of failing of the components on the $k_{\mathbf{x}}!$ different ways with equal probabilities, using (3.7), (3.8) and (4.9) we obtain

$$E(U_{\mathbf{x}}) = \frac{1}{\lambda}\left(\frac{n_{\mathbf{x}}p_{\mathbf{x}}}{n-k_{\mathbf{x}}} - A(\mathbf{x})\sum_{j_1=a(1)}^{b(1)}\sum_{j_2=a(2)}^{b(2)}\cdots\sum_{j_{k_{\mathbf{x}}}=a(k_{\mathbf{x}})}^{b(k_{\mathbf{x}})}\sum_{s=1}^{k_{\mathbf{x}}}\frac{j_s\prod_{s=1}^{k_{\mathbf{x}}}B(s,j_s)}{(n-s)(n-s+1)}\right)$$
(4.12)

where $A(\mathbf{x})$, $B(s,j_s)$, $a(s)$ and $b(s)$ are given by (3.6) and $p_{\mathbf{x}}$ is given by (3.5). Putting (4.11) and (4.12) in (4.7) and (4.5) we prove the theorem. $\qquad\square$

In order to construct an algorithm, $D_{\mathbf{x}}$ can be written as

$$D_{\mathbf{x}} = \frac{n_{\mathbf{x}}p_{\mathbf{x}}}{n-k_{\mathbf{x}}} - A(\mathbf{x})\sum_{l=1}^{k_{\mathbf{x}}}\frac{1}{(n-l)(n-l+1)}H(l),$$
(4.13)

where

$$H(l) = \sum_{j_1=a(1)}^{b(1)}\widetilde{B}(l,1,j_1)\sum_{j_2=a(2)}^{b(2)}\widetilde{B}(l,2,j_2)\ldots\sum_{j_{k_{\mathbf{x}}}=a(k_{\mathbf{x}})}^{b(k_{\mathbf{x}})}\widetilde{B}(l,k_{\mathbf{x}},j_{k_{\mathbf{x}}}),$$
(4.14)

$$\widetilde{B}(l,s,j_s) = \begin{cases} B(s,j_s), & l \neq s, \\ j_sB(s,j_s), & l = s, \end{cases}$$

and $A(\mathbf{x})$ is given by (3.6).

In the algorithm for the estimation of $\widehat{\lambda}$ we use all functions defined in the previous section, so we will give only the next steps.

1. Define a function $A_1(\mathbf{x}) = \dfrac{n_{\mathbf{x}}p_{\mathbf{x}}}{n-k_{\mathbf{x}}}$;

2. Define a function $\widetilde{B}(l,s,j_s) = \begin{cases} B(s,j_s), & l \neq s, \\ j_sB(s,j_s), & l = s; \end{cases}$

3. Define $L(\mathbf{x},l,s,j) = \begin{cases} \widetilde{B}(l,s,j), & s = k_{\mathbf{x}}, \\ \widetilde{B}(l,s,j)\displaystyle\sum_{u=\text{Max}((s+1)M,j+1)}^{n_{\mathbf{c}}-k_{\mathbf{c}}+s+1}\widetilde{L}(\mathbf{c},l,s+1,u), & s < k_{\mathbf{c}}; \end{cases}$

4. $V_1(\mathbf{x},l) = \displaystyle\sum_{u=M}^{n_{\mathbf{x}}-k_{\mathbf{x}}+1}\widetilde{L}(\mathbf{x},l,1,u)$ and $V_2(\mathbf{x}) = \displaystyle\sum_{l=1}^{k_{\mathbf{x}}}\frac{1}{(n-l)(n-l+1)}V_1(\mathbf{x},l)$;

5. $D(\mathbf{x}) = \begin{cases} A_1(\mathbf{x}), & k_{\mathbf{x}} = 0, \\ A_1(\mathbf{x}) - A(\mathbf{x})V_2(\mathbf{x}), & k_{\mathbf{x}} \neq 0; \end{cases}$

6. $E(\mathbf{c}) = \displaystyle\sum_{\mathbf{x}\in S\backslash\mathcal{C}_f,\mathbf{x}=\mathbf{c}+\mathbf{e}_i}\frac{1}{n-k_{\mathbf{x}}}\left(D(\mathbf{x})+\frac{p_{\mathbf{x}}}{n-k_{\mathbf{x}}}\right),\ \mathbf{c}\in\mathcal{C}_f$;

7. $\widehat{\lambda} = \dfrac{N}{\sum_{i=1}^{N}t_i}\displaystyle\sum_{\mathbf{c}\in\Gamma}E(\mathbf{c})$.

5. Reliability function

In order to find the reliability function for this type of system we define a function

$$H_{h_1,h_2,\ldots,h_s}(\lambda,t) = P(T_1 + T_2 + \ldots + T_s < t), \tag{5.1}$$

where h_i are integers that satisfy $h_1 \leq h_2 \leq \ldots \leq h_s$, $T_1 \sim Gamma(h_1, n\lambda)$ and $T_i \sim Gamma(h_i - h_{i-1}, (n - i + 1)\lambda)$, $i = \overline{2, s}$.

Theorem 5.1. *The reliability function of a MMS system in which all transition intensities are equal is given by*

$$R(t) = 1 - \sum_{\mathbf{c} \in \mathcal{C}_f} \sum_{\mathbf{x} \in S \backslash \mathcal{C}_f, \ \mathbf{x} = \mathbf{c} + \mathbf{e}_i} D_{\mathbf{x}}(t), \tag{5.2}$$

where

$$D_{\mathbf{x}}(t) = \begin{cases} H_{n_{\mathbf{c}}}(\lambda, t) \left(\dfrac{1}{n}\right)^{n_{\mathbf{c}}} \dfrac{(\sum_{i=1}^n \widetilde{x}_i)!}{\prod_{i=1}^n \widetilde{x}_i!}, & k_{\mathbf{x}} = 0, \\ \dfrac{A(\mathbf{x})}{n - k_{\mathbf{x}}} \displaystyle\sum_{j_1 = a(1)}^{b(1)} \sum_{j_2 = a(2)}^{b(2)} \cdots \sum_{j_{k_{\mathbf{x}}} = a(k_{\mathbf{x}})}^{b(k_{\mathbf{x}})} B(s, j_s) H_{j_1, j_2, \ldots, j_{k_{\mathbf{x}}}, n_{\mathbf{c}}}(\lambda, t), & k_{\mathbf{x}} \neq 0, \end{cases}$$

and $A(\mathbf{x})$, $B(s, j_s)$, $a(1)$, $a(s)$ *and* $b(s)$ *are given by (3.6).*

Proof. Refer to (4.2).

$$\begin{aligned} F(t) & = P(T < t) = \sum_{\gamma \in \Gamma} P(T_\gamma < t)\widetilde{p}_\gamma = \sum_{\mathbf{c} \in \mathcal{C}_f} \sum_{\gamma \in \Gamma(\mathbf{c})} P(T_\gamma < t)\widetilde{p}_\gamma \\ & = \sum_{\mathbf{c} \in \mathcal{C}_f} \sum_{\mathbf{x} \in S \backslash \mathcal{C}_f, \mathbf{x} = \mathbf{c} + \mathbf{e}_i} \sum_{\gamma \in \Gamma(\mathbf{x})} P(T_\gamma + T_{\mathbf{x},\mathbf{c}} < t)\widetilde{p}_\gamma p_{\mathbf{x},\mathbf{c}} \\ & = \sum_{\mathbf{c} \in \mathcal{C}_f} \sum_{\mathbf{x} \in S \backslash \mathcal{C}_f, \mathbf{x} = \mathbf{c} + \mathbf{e}_i} D_{\mathbf{x}}(t). \end{aligned} \tag{5.3}$$

In the case when there are no zero coordinates in $\mathbf{x}$, i.e. $k_{\mathbf{x}} = 0$, $P(T_{\gamma_{\mathbf{x}}} + T_{\mathbf{x},\mathbf{c}} < t) = H_{n_{\mathbf{c}}}(\lambda, t)$, $\widetilde{p}_{\gamma_{\mathbf{x}}} p_{\mathbf{x},\mathbf{c}} = \left(\frac{1}{n}\right)^{n_{\mathbf{c}}}$ and the number of paths to $\mathbf{x}$ is $\frac{(\sum_{i=1}^n \widetilde{x}_i)!}{\prod_{i=1}^n \widetilde{x}_i!}$, so

$$D_{\mathbf{x}}(t) = H_{n_{\mathbf{c}}}(\lambda, t) \left(\frac{1}{n}\right)^{n_{\mathbf{c}}} \frac{(\sum_{i=1}^n \widetilde{x}_i)!}{\prod_{i=1}^n \widetilde{x}_i!}.$$

Consider the case when $\mathbf{x}$ has some coordinates equal to 0. Let the path $\gamma_{\mathbf{x}}$ be such that the total failure of its components are at the steps j_s, $s = \overline{1, k_{\mathbf{x}}}$. Now

$$P(T_{\gamma_{\mathbf{x}}} + T_{\mathbf{x},\mathbf{c}} < t) = H_{j_1, j_2, \ldots, j_{k_{\mathbf{x}}}, n_{\mathbf{c}}}(\lambda, t)$$

and from (3.9) we have that

$$\widetilde{p}_{\gamma_{\mathbf{x}}} p_{\mathbf{x},\mathbf{c}} = \left(\frac{1}{n - k_{\mathbf{x}}}\right)^{n_{\mathbf{c}}} \prod_{s=0}^{k_{\mathbf{x}}} \left(\frac{n - s}{n - s + 1}\right)^{j_s}.$$

Using the number of ways in which we can order the numbers j_s and the number of paths for chosen numbers j_s, $s = \overline{1, k_{\mathbf{x}}}$, on similar way as in Theorem 3.1, we can obtain that

$$D_{\mathbf{x}}(t) = \frac{A(\mathbf{x})}{n - k_{\mathbf{x}}} \sum_{j_1=a(1)}^{b(1)} \sum_{j_2=a(2)}^{b(2)} \cdots \sum_{j_{k_{\mathbf{x}}}=a(k_{\mathbf{x}})}^{b(k_{\mathbf{x}})} B(s, j_s) H_{j_1, j_2, \ldots, j_{k_{\mathbf{x}}}, n_{\mathbf{c}}}(\lambda, t),$$

where $A(\mathbf{x})$, $B(s, j_s)$, $a(1)$, $a(s)$ and $b(s)$ are given by (3.6). $\qquad \square$

References

[1] M.T. Abuelma'atti and I.S. Qamber, *Reliability Analysis of a Redundant System with Non-repairable Units and Common-Cause Failures Using the SPICE Circuit Simulation Program*, Proc. Natl. Sci. Counc. ROC(A) **25** (no. 5) (2001), 322–328.

[2] M.J. Crowder, A.C. Kimber, R.L. Smith and T.J. Sweeting, *Statistical Analysis of Reliability Data*, Chapman and Hall, London, 1991.

[3] A. Dharmadhikari, S.B. Kulathinal and V. Mandrekar, *An algorithm for the estimation of minimal cut and ath sets from field failure data*, Statist. Probab. Lett. **58** (no. 1) (2002), 1–11.

[4] B. Dimitrov and V. Rujkov, *On Multi-state reliability system*, Informacionniue processu, Tom2, No 2, 2002 246-251.

[5] B. Dimitrov, V. Rujkov and P. Stanchev, *On multi-state reliability systems*, in: Third international conference on Mathematical Methods in Reliability, MMR 2002, Trondheim, Norway, 201–204.

[6] J. Huang, M.J. Zuo and Y. Wu, *Generalized Multi-State k-out-of-n: G Systems*, IEEE Transactions on Reliability **49** (no. 1) (2000), 105–111.

[7] B.H. Lindqwist, *Bounds for the reliability of the multi-state system with partially ordered state spaces and stochastically monotone Markov transitions*, preprint no. 11, Norwegian University of Science and Technology, Trondheim, Norway, 2002.

Marija Mihova and Zaneta Popeska
Faculty of Natural Sciences and Mathematics
Gazibaba bb, P.O. Box 162
1000 Skopje
Macedonia
e-mail: `marija@ii.edu.mk`
      `zaneta@ii.edu.mk`