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On Recursive Derivates of k-ary Operations

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Abstract

We present several results about recursive derivates of k-ary operations defined on a finite set Q. They are generalizations of some binary cases given by Larionova-Cojocaru and Syrbu [5]. Also, we present several experimental results about recursive differentiability of ternary quasigroups of order 4.

Keywords: recursively differentiable quasigroups, orthogonality

1 Introduction

Let Q be a nonempty set and let k be a positive integer. We will use (x_1^k) to denote the k-tuple $(x_1, \ldots, x_k) \in Q^k$. A *$k-$ary operation* f on the set Q is a mapping $f : Q^k \to Q$ defined by $f : (x_1^k) \to x_{k+1}$, for which we write $f(x_1^k) = x_{k+1}$. A k-ary groupoid ($k \geq 1$) is an algebra (Q, f) on a nonempty set Q as its universe and with one k-ary operation f. A k-ary groupoid (Q, f) is called a *k-ary quasigroup* (of order $|Q| = q$) if any k of the elements $a_1, a_2, \ldots, a_{k+1} \in Q$, satisfying the equality $f(a_1^k) = a_{k+1}$, uniquely specifies the remaining one.

The $k-$ary operations $f_1, f_2, \ldots, f_d, 1 \leq d \leq k$, defined on a set Q are **orthogonal** if the system $\{f_i(x_1^k) = a_i\}_{i=1}^d$ has exactly q^{k-d} solutions for any $a_1, \ldots, a_d \in Q$, where $q = |Q|$ [2]. There is an one-to-one correspondence between the set of all k-tuples of orthogonal k-ary operations $< f_1, f_2, \ldots, f_k >$ defined on a set Q and the set of all permutations $\theta : Q^k \to Q^k$ ([2]), given by

$$\theta(x_1^k) \to (f_1(x_1^k), f_2(x_1^k), \ldots, f_d(x_1^k)).$$

The k-ary operation I_j, $1 \le j \le k$, defined on Q with $I_j(x_1^k) = x_j$ is called the *j-th selector* or the *j-th projection*.

A system $\Sigma = \{f_1, f_2, \ldots, f_s\}_{s \ge k}$ of k-ary operations is called **orthogonal**, if every k operations of Σ are orthogonal. A system $\Sigma = \{f_1, f_2, \ldots, f_r\}, r \ge 1$ of distinct $k-$ary operations defined on a set Q is called **strong orthogonal** if the system $\{I_1, \ldots, I_k, f_1, f_2, \ldots, f_r\}$ is orthogonal, where each $I_j, 1 \le j \le k$, is $j-$th selector. It follows that each operation of a strong orthogonal system, which is not a selector, is a k-ary quasigroup operation.

A code $C \subseteq Q^n$ is called a **complete k-recursive code** if there exists a function $f : Q^k \to Q$ $(1 \le k \le n)$ such that every code word $(u_0, \ldots, u_{n-1}) \in C$ satisfies the conditions $u_{i+k} = f(u_i^{i+k-1})$ for every $i = 0, 1, \ldots, n-k-1$, where $u_0, \ldots, u_{k-1} \in Q$. It is denoted by $C(n, f)$.

$C(n, f)$ can be represented by

$$C(n, f) = \{(x_1^k, f^{(0)}(x_1^k), \ldots, f^{(n-k-1)}(x_1^k)) : (x_1^k) \in Q^k\}$$

where $f^{(0)} = f^{(0)}(x_1^k) = f(x_1^k)$,
$f^{(1)} = f^{(1)}(x_1^k) = f(x_2^k, f^{(0)})$
...
$f^{(k-1)} = f^{(k-1)}(x_1^k) = f(x_k, f^{(0)}, \ldots, f^{(k-2)})$
$f^{(i+k)} = f^{(i+k)}(x_1^k) = f(f^{(i)}, \ldots, f^{(i+k-1)})$ for $i \ge 0$
are **recursive derivatives** of f. The general form of the recursive derivatives for any k-ary operation f is given in [4], and $f^{(n)} = f\theta^n$, where $\theta : Q^k \to Q^k, \theta(x_1^k) = (x_2^k, f(x_1^k))$.

A k-quasigroup (Q, f) is called **recursively t-differentiable** if all its recursive derivatives $f^{(0)}, \ldots, f^{(t)}$ are k-ary quasigroup operations [3]. A k-quasigroup (Q, f) is called **t-stable** if the system of all recursive derivatives $f^{(0)} \ldots, f^{(t)}$ of f is an orthogonal system of k-ary quasigroup operations, i.e. $C(k + t + 1, f)$ is an MDS code [3]. A k-ary quasigroup (Q, f) is called **strongly recursively t-differentiable** if it is recursively t-differentiable and $f^{(t+1)} = I_1$ (introduced for binary case in [1]). A k-ary quasigroup (Q, f) is strongly recursively 0-differentiable if $f^{(1)} = I_1$.

2 Main results

The following results are generalisation of binary cases for recursive derivates from [5].

Proposition 1. *Let (Q, f) be a k-ary groupoid. For every $(x_1^k) \in Q^k$ the following equalities hold:*

$$f^{(n)}(x_1^k) = f^{(n-1)}(x_2^k, f^{(0)}(x_1^k)), \forall n \in N$$

Proposition 2. *Let (Q, f) be a k-ary groupoid. For every $(x_1^k) \in Q^k$ and for every $j = k-1, \ldots, n-1$, where $n \geq k$, the following equalities hold:*

$$f^{(n)}(x_1^k) = f^{(n-j-1)}(f^{(j-k+1)}(x_1^k), \ldots, f^{(j)}(x_1^k))$$

Proposition 3. *If two k-ary groupoids (Q_1, f) and (Q_2, g) are isomorphic, then their recursive derivatives $(Q_1, f^{(n)})$ and $(Q_2, g^{(n)})$ are isomorphic too, for every $n \geq 1$.*

Proposition 4. *If (Q, f) is a k-ary groupoid, then $Aut(Q, f)$ is a subgroup of $Aut(Q, f^{(n)})$, for every $n \geq 1$.*

3 Experimental results for ternary quasigroups of order 4

By experiments, we obtained the following results:

- there are 96 recursively 1-differentiable ternary quasigroups of order 4, and all are 1-stable
- there are no recursively t-differentiable ternary quasigroups of order 4, for $t \geq 2$,
- there are 64 strongly recursively 0-differentiable ternary quasigroups of order 4,
- there are 8 strongly recursively 1-differentiable ternary quasigroups of order 4.

Bellow is an example of strongly recursively 1-differentiable and 1-stable ternary quasigroups of order 4.

$$\{\{\{1,2,3,4\},\{3,4,1,2\},\{4,3,2,1\},\{2,1,4,3\}\},\{\{2,1,4,3\},\{4,3,2,1\},\{3,4,1,2\},\{1,2,3,4\}\},$$
$$\{\{3,4,1,2\},\{1,2,3,4\},\{2,1,4,3\},\{4,3,2,1\}\},\{\{4,3,2,1\},\{2,1,4,3\},\{1,2,3,4\},\{3,4,1,2\}\}\}$$

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