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A new method for computing the number of n-quasigroups

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Abstract. We use the isotopy classes of quasigroups for computing the numbers of finite n-quasigroups ($n = 1, 2, 3, \dots$). The computation is based on the property that every two isotopic n-quasigroups are substructures of the same number of $n+1$-quasigroups. This is a new method for computing the number of n-quasigroups and in an enough easy way we could compute the numbers of ternary quasigroups of orders up to and including 5 and of quaternary quasigroups of orders up to and including 4.

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1 Introduction

An n-groupoid ($n \geq 1$) is an algebra (Q, f) on a nonempty set Q as its universe and with one n-ary operation $f : Q^n \to Q$. An n-groupoid (Q, f) is said to be an n-quasigroup if any n of the elements $a_1, a_2, \dots, a_{n+1} \in Q$, satisfying the equality

$$f(a_1, a_2, \dots, a_n) = a_{n+1},$$

uniquely determine the other one [1]. An n-groupoid is said to be a cancellative n-groupoid if it satisfies the cancellation law

$$f(a_1, \dots, a_i, x, a_{i+2}, \dots, a_n) = f(a_1, \dots, a_i, y, a_{i+2}, \dots, a_n) \Rightarrow x = y$$

for each $i = 0, 1, \dots, n-1$ and every $a_j \in Q$. An n-groupoid is said to be a solvable n-groupoid if the equation $f(a_1, \dots, a_i, x, a_{i+2}, \dots, a_n) = a_{n+1}$ has a solution x for each $i = 0, 1, \dots, n-1$ and every $a_j \in Q$.

The definition of an n-quasigroup immediately implies the following.

Lemma 1. *Let (Q, f) be a finite n-quasigroup and let the mapping $\varphi : Q \to Q$ be defined by $\varphi(x) = f(a_1, \dots, a_i, x, a_{i+2}, \dots, a_n)$. Then φ is a permutation on Q.*

Here we consider only finite n-quasigroups (Q, f), i.e., Q is a finite set, and in this case we have the next property.

Proposition 1. *The following statements for a finite n-groupoid (Q, f) are equivalent:*

(a) (Q, f) *is an n-quasigroup.*
(b) (Q, f) *is a cancellative n-groupoid.*
(c) (Q, f) *is a solvable n-groupoid.*

Proof. $(a) \Rightarrow (b)$ follows immediately by the definitions.

$(a) \Rightarrow (c)$ follows by Lemma 1.

Clearly, (b) and (c) imply (a).

$(b) \Rightarrow (c)$: Let (Q, f) be a cancellative n-groupoid. Then

$$\{f(a_1, \dots, a_i, x, a_{i+2}, \dots, a_n) |\ x \in Q\} = Q$$

for any fixed $a_j \in Q$.

$(c) \Rightarrow (b)$: If the groupoid (Q, f) is not cancellative then, for some $a_j \in Q$ and $i \in \{0, \dots, n-1\}$, the equation $f(a_1, \dots, a_i, x, a_{i+2}, \dots, a_n) = a_{n+1}$ has two different solutions $x_1 \neq x_2$. Then there is an element $b \in Q$ such that $b \notin \{f(a_1, \dots, a_i, x, a_{i+2}, \dots, a_n) |\ x \in Q\}$. Hence, the equation $f(a_1, \dots, a_i, x, a_{i+2}, \dots, a_n) = b$ has no solution on x. $\square$

Given n-quasigroups (Q, f) and (Q, h), we say that (Q, f) is isotopic to (Q, h) if there are permutations $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ on Q such that for every $a_j \in Q$

$$\alpha_{n+1} f(a_1, \dots, a_n) = h(\alpha_1 a_1, \dots, \alpha_n a_n).$$

If (Q, f) is isotopic to (Q, h), then (Q, h) is isotopic to (Q, f) too, since for the permutations $\alpha_1^{-1}, \alpha_2^{-1}, \dots, \alpha_{n+1}^{-1}$ we have $\alpha_{n+1}^{-1} h(a_1, a_2, \dots, a_n) = f(\alpha_1^{-1} a_1, \dots \dots, \alpha_n^{-1} a_n)$. Then we say that the $n+1$-tuple of permutations $(\alpha_1, \dots, \alpha_{n+1})$ is an isotopism between the n-quasigroups (Q, f) and (Q, h). The set of all isotopisms of an n-quasigroup is a group under the operation [3]:

$$(\alpha_1, \alpha_2, \dots, \alpha_{n+1})(\beta_1, \beta_2, \dots, \beta_{n+1}) = (\alpha_1\beta_1, \alpha_2\beta_2, \dots, \alpha_{n+1}\beta_{n+1}).$$

Also, the relation "is isotopic to" is an equivalence relation in the set of all n-quasigroups over a set Q. The equivalence classes are called the classes of isotopism or isotopy classes.

Example 1. A unary quasigroup (Q, f) is in fact a permutation on the set Q. If (Q, f) and (Q, g) are unary quasigroups, then they are isotopic by the isotopism (g^{-1}, f^{-1}). Hence, there is only one isotopy class in the set of unary quasigroups over given universe.

Let $Q = \{1, 2, \dots, r\},\ r > 0$. An $\underbrace{r \times \cdots \times r}_{n}$-matrix $L = [l_{i_1, i_2, \dots, i_n}]$, such that for each $i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_n$ and each j the $(i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_n)$-th row vector $(l_{i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_n}, l_{i_1,\dots,i_{j-1},2,i_{j+1},\dots,i_n}, \dots, l_{i_1,\dots,i_{j-1},r,i_{j+1},\dots,i_n})$ of L is a permutation of Q, is said to be an n-Latin square of order r. The main body of the multiplication table of an n-quasigroup (Q, f) is an n-Latin square. Conversely, from an n-Latin square we can obtain an n-quasigroup, by its bordering [2, 5]. (Note that a 1-Latin square is a permutation of Q, and a 2-Latin square is a Latin square [2].)

In this paper we give a new method for computing the number of n-quasigroups, that is based on the main theorem from Section 2. For computing the number of $n+1$-quasigroups one needs the number of elements of each isotopy class of n-quasigroups, and a representative of each isotopy class. We note that the other methods for computing the number of $n+1$-quasigroups of order r use the formula $L_r = r!(r-1)!^n N_r$, where L_r is the number of all $n+1$-quasigroups of order r, and N_r is the number of so called normal $n+1$ quasigroups of order r. Usually, the number N_r is computed by different combinatorial technique, while our approach is algebraically based.

Applications of our method for computing the number of n-quasigroups are given in Section 3. For that aim we introduce a linear ordering of the set of n-quasigroups on the universe set $\{1, 2, \ldots, r\}$. The obtained results are the same as those obtained by other methods.

2 Main theorem

The problem of enumerating the set of quasigroups of given order r is well known. In fact, only the number of binary quasigroups of order $r \le 11$ is known [4]. Nowadays, one can handle by personal computer only the set of quasigroups of order $r \le 6$ (or maybe 7), since there are about 8.12×10^8 quasigroups of order 6, 6.14×10^{13} quasigroups of order 7 and 1.08×10^{20} quasigroups of order 8.

The main theorem of this paper allows the numbers of $n+1$-quasigroups (of small orders) to be computed, provided the isotopy classes of n-quasigroups of given order are known.

Given an $n+1$-quasigroup (Q, f) of order $r = |Q|$, $n \ge 1$, we define an (a,i)-projected n-quasigroup $(Q, f_{a,i})$ for each $i = 1, 2, \ldots, n+1$ and each $a \in Q$ by

$$f_{a,i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) := f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_{n+1}).$$

We have by Proposition 1 that $f_{a,i}$ is an n-quasigroup operation and that

$$f_{a,i} = f_{b,i} \iff a = b. \tag{1}$$

This implies that the $n+1$-ary operation f is uniquely determined by each of the sets $F_i = \{f_{a,i} | \ a \in Q\}$ of (a,i)-projected n-ary operations ($i = 1, 2, \ldots, n+1$).

Proposition 2. *Let Q be a finite nonempty set and let $\{f_a | \ a \in Q\}$ be a set of n-quasigroup operations on Q such that*

$$a \ne b \Rightarrow f_a(a_1, \ldots, a_n) \ne f_b(a_1, \ldots, a_n) \tag{2}$$

for every $a_1, \ldots, a_n \in Q$. Fix a number $i \in \{1, 2, \ldots, n+1\}$. Then an $n+1$-quasigroup (Q, f) can be defined such that (Q, f_a) are its (a,i)-projected n-quasigroups, i.e. $(Q, f_{a,i}) = (Q, f_a)$ for each $a \in Q$.

Proof. Choose a number $i \in \{1, 2, \ldots, n+1\}$ and define an $n+1$-ary operation f by

$$f(a_1, a_2, \ldots, a_{n+1}) := f_{a_i}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n+1})$$

for every $a_1, \ldots, a_{n+1} \in Q$. Then (Q, f) is a cancellative $n+1$-groupoid, hence it is an $n+1$-quasigroup by Proposition 1. By the definition of an (a,i)-projected n-quasigroup, we have $f_{a,i} = f_a$. $\square$

Theorem 1. *Let $Q = \{q_1, q_2, \dots, q_r\}$, $r \geq 1$, and let (Q, g) and (Q, h) be two n-quasigroups from the same isotopy class. Fix a number $i \in \{1, 2, \dots, n+1\}$. Then the number of $n+1$-quasigroups having (Q, g) as its (q_1, i)-projected n-quasigroup is equal to the number of $n+1$-quasigroups having (Q, h) as its (q_1, i)-projected n-quasigroup.*

Proof. Fix a number $i \in \{1, 2, \dots, n+1\}$. Let $(\alpha_1, \alpha_2, \dots, \alpha_{n+1})$ be an isotopism from (Q, g) to (Q, h), i.e.

$$\alpha_{n+1} g(a_1, \dots, a_n) = h(\alpha_1 a_1, \dots, \alpha_n a_n)$$

for each $a_1, \dots, a_n \in Q$. Let (Q, f) be an $n+1$-quasigroup such that $f_{q_1,i} = h$. Then, for the projected quasigroups, by (1) we have

$$f_{q_s,i} = f_{q_t,i} \Longleftrightarrow s = t. \tag{3}$$

Define a set of n-quasigroups $\{(Q, f'_q) |\ q \in Q\}$ by $f'_{q_1} = g$ and

$$f'_{q_j}(x_1, x_2, \dots, x_n) := \alpha_{n+1}^{-1} f_{q_j,i}(\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n) \tag{4}$$

for $j = 2, 3, \dots, r$.

The condition (2) of Proposition 2 is satisfied for the set of n-quasigroups $\{(Q, f'_q) |\ q \in Q\}$. Namely, if $f'_{q_s} = f'_{q_t}$, then

$$\alpha_{n+1}^{-1} f_{q_s,i}(\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n) = \alpha_{n+1}^{-1} f_{q_t,i}(\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n)$$

and that implies

$$f_{q_s,i}(\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n) = f_{q_t,i}(\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n).$$

Since α_k are permutations, we have $f_{q_s,i} = f_{q_t,i}$, leading to $s = t$ by (3). Now, by Proposition 2, we can define an $n+1$-quasigroup (Q, f') such that $f'_{q_1,i} = g$ and $f'_{q_j,i} = f'_{q_j}$ for $j \geq 2$.

We showed that to any $n+1$-quasigroup (Q, f) satisfying the condition $f_{q_1,i} = h$ we can adjoin an $n+1$-quasigroup (Q, f') satisfying the condition $f'_{q_1,i} = g$. If $(Q, \tilde{f})$ is another $n+1$-quasigroup satisfying the condition $\tilde{f}_{q_1,i} = h$ and if an $n+1$-quasigroup $(Q, \tilde{f}')$ is constructed from $\tilde{f}$ as above, then $\tilde{f}' \neq f'$. Namely, the equality $\tilde{f}' = f'$ implies, by (4),

$$\alpha_{n+1}^{-1} f_{q_j,i}(\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n) = \alpha_{n+1}^{-1} \tilde{f}_{q_j,i}(\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n),$$

i.e. we have $f_{q_j,i} = \tilde{f}_{q_j,i}$ for each $j = 1, 2, \dots, r$. Hence, $f = \tilde{f}$.

We proved that the number of $n+1$-quasigroups having (Q, g) as its (q_1, i)-projected n-quasigroup is not smaller than the number of $n+1$-quasigroups having (Q, h) as its (q_1, i)-projected n-quasigroup. Analogously, the number of $n+1$-quasigroups having (Q, h) as its (q_1, i)-projected n-quasigroup is not smaller than the number of $n+1$-quasigroups having (Q, g) as its (q_1, i)-projected n-quasigroup. $\square$

Corollary 1. *Let $Q = \{q_1, q_2, \dots, q_r\}$, $r \geq 1$, and let the isotopy classes of the n-quasigroups on Q be $C_1, C_2, \dots, C_k$. Then the number of $n+1$-quasigroups on Q is equal to*

$$b_1|C_1| + b_2|C_2| + \cdots + b_k|C_k| \tag{5}$$

where b_i denotes the number of $n+1$-quasigroups having as its $(q_1, 1)$-projected n-quasigroup an n-quasigroup from the class C_i.

Example 2. There are 6 unary quasigroups on the set $Q = \{1, 2, 3\}$ and they can be represented as the permutations 123, 132, 213, 231, 312 and 321. They form one class of isotopism C_1 and the unary quasigroup 123 can be (1,1)-projected quasigroup to $b_1 = 2$ binary quasigroups:

$*_1$	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2

$*_2$	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

Consequently, there are $2 \times 6 = 12$ binary quasigroups on the set $\{1, 2, 3\}$.

3 Numerical results

The main theorem of this paper helps us to compute the numbers of n-quasigroups of order r. We could do that only for smaller values of r. For computing purposes we present the set of n-quasigroups of order r linearly and we order them lexicographically as follows. We take that the universe set is $Q = \{1, 2, \dots, r\}$ and that the n-quasigroups are given by their n-Latin squares. The unary quasigroups are linearly presented and lexicographically ordered in a natural way, since its 1-Latin square consist of only one permutation of Q. An $n+1$-quasigroup (Q, f) of order r is uniquely determined by its (q, i)-projected quasigroups $(Q, f_{1,i}), (Q, f_{2,i}), \dots, (Q, f_{r,i})$, for each fixed $i \in \{1, 2, \dots, n+1\}$. We fix $i = 1$ and let $S_1, S_2, \dots, S_r$ be the linear presentations of the quasigroups $(Q, f_{1,1}), (Q, f_{2,1}), \dots, (Q, f_{r,1})$ respectively. Then the linear presentation of the $n+1$-quasigroup (Q, f) is given by

$$S_1 \underbrace{||\dots|}_{n} S_2 \underbrace{||\dots|}_{n} \dots \underbrace{||\dots|}_{n} S_r. \tag{6}$$

Now, the lexicographic ordering of the linear presentations of all n-quasigroups of order r gives the ordering of the quasigroups.

Example 3. On the set $\{1, 2, 3\}$ we have the following linear presentations and lexicographically ordering of the unary, binary and ternary quasigroups.

$123 < 132 < 213 < 231 < 312 < 321$,

$123|231|312 < 123|312|231 < 132|213|321 < 132|321|213 <$

$< 213|132|321 < 213|321|132 < 231|123|312 < 231|312|123 <$
$< 312|123|231 < 312|231|123 < 321|132|213 < 321|213|132,$

$$
\begin{aligned}
&123|231|312||231|312|123||312|123|231 < 123|231|312||312|123|231||231|312|123 <\\
&< 123|312|231||231|123|312||312|231|123 < 123|312|231||312|231|123||231|123|312 <\\
&< 132|213|321||213|321|132||321|132|213 < 132|213|321||321|132|213||213|321|132 <\\
&< 132|321|213||213|132|321||321|213|132 < 132|321|213||321|213|132||213|132|321 <\\
&< 213|132|321||132|321|213||321|213|132 < 213|132|321||321|213|132||132|321|213 <\\
&< 213|321|132||132|213|321||321|132|213 < 213|321|132||321|132|213||132|213|321 <\\
&< 231|123|312||123|312|231||312|231|123 < 231|123|312||312|231|123||123|312|231 <\\
&< 231|312|123||123|231|312||312|123|231 < 231|312|123||312|123|231||123|231|312 <\\
&< 312|123|231||123|231|312||231|312|123 < 312|123|231||231|312|123||123|231|312 <\\
&< 312|231|123||123|312|231||231|123|312 < 312|231|123||231|123|312||123|312|231 <\\
&< 321|132|213||132|213|321||213|321|132 < 321|132|213||213|321|132||132|213|321 <\\
&< 321|213|132||132|321|213||213|132|321 < 321|213|132||213|132|321||132|321|213.
\end{aligned}
$$

Thus, on the set $\{1, 2, 3\}$, the 5-th binary quasigroup is 213|132|321 and the 16-th ternary quasigroup is 231|312|123||312|123|231||123|231|312. One can see that the 14-th ternary quasigroup is built up from the 8-th, 9-th and the first binary quasigroups. □

Trivially, there is only one n-quasigroup of order 1 and $r!$ unary quasigroups of order r. For computing the number of n-quasigroups of order 2 and 3 it is useful to be noted the following. Let in (6) us $S_1 = s_{11}s_{12}\dots s_{1r}|t_{11}t_{12}\dots t_{1r}|\dots$, $S_2 = s_{21}s_{22}\dots s_{2r}|t_{21}t_{22}\dots t_{2r}|\dots$, $S_r = s_{r1}s_{r2}\dots s_{rr}|t_{r1}t_{r2}\dots t_{rr}|\dots$, where $s_{\lambda\mu}, t_{\lambda\mu} \in \{1, 2, \dots, r\}$. Then

$$
\begin{array}{c}
s_{11}s_{21}\dots s_{r1}, \quad s_{12}s_{22}\dots s_{r2}, \dots, \quad s_{1r}s_{2r}\dots s_{rr},\\
t_{11}t_{21}\dots t_{r1}, \quad t_{12}s_{22}\dots t_{r2}, \dots, \quad t_{1r}t_{2r}\dots t_{rr}, \quad \dots
\end{array} \tag{7}
$$

are permutations of $\{1, 2, \dots, r\}$. Immediately we have:

Proposition 3. *There are only* 2 *n-quasigroups of order* 2.

Proposition 4. *The number of n-quasigroups of order* 3 *is* 3×2^n.

Proof. Let $S_1 \underbrace{||\dots|}_{n} S_2 \underbrace{||\dots|}_{n} S_3$ be an $n+1$-quasigroup of order 3. S_1 can be any n-quasigroup of order 3. Given S_1, by (7), there are only two choices for S_2; given S_1 and S_2, the quasigroup S_3 is uniquely determined. Since we have 6 1-quasigroups of order 3, the result follows. □

Proposition 5. *The number of n-quasigroups of order* 4 *for* $n = 1$, $n = 2$, $n = 3$ *and* $n = 4$ *are* 24, 576, 55 296 *and* 36 972 288 *respectively.*

Isotopy class	Represent of C_i	$\|C_i\|$	b_i	$b_i\|C_i\|$
C_1	1234|2143|3412|4321|| 2143|1234|4321|3412|| 3412|4321|1234|2143|| 4321|3412|2143|1234	**864**	2292	1980288
C_2	1234|2143|3421|4312|| 2143|1234|4312|3421|| 3421|4312|2143|1234|| 4312|3421|1234|2143	**2592**	852	2208384
C_3	1234|2143|3412|4321|| 2143|1234|4321|3412|| 3412|4321|2143|1234|| 4321|3412|1234|2143	**2592**	876	2270592
C_4	1234|2143|3412|4321|| 2143|1234|4321|3412|| 3421|4312|1243|2134|| 4312|3421|2134|1243	**2592**	876	2270592
C_5	1234|2143|3412|4321|| 2143|1234|4321|3412|| 3421|4312|2134|1243|| 4312|3421|1243|2134	**2592**	876	2270592
C_6	1432|3241|4123|2314|| 4123|2314|1432|3241|| 3214|4132|2341|1423|| 2341|1423|3214|4132	**2592**	876	2270592
C_7	1432|3241|4123|2314|| 4123|2314|1432|3241|| 3241|1432|2314|4123|| 2314|4123|3241|1432	**2592**	876	2270592
C_8	1432|3241|4123|2314|| 4123|2314|1432|3241|| 3214|1423|2341|4132|| 2341|4132|3214|1423	**2592**	876	2270592
C_9	1234|2341|3412|4123|| 4123|3412|2341|1234|| 3412|1234|4123|2341|| 2341|4123|1234|3412	**5184**	144	746496
C_{10}	1234|2341|3412|4123|| 4321|1432|2143|3214|| 2413|3124|4231|1342|| 3142|4213|1324|2431	**5184**	144	746496
C_{11}	1243|2431|3124|4312|| 3421|4213|1342|2134|| 2314|3142|4231|1423|| 4132|1324|2413|3241	**5184**	144	746496
C_{12}	1234|2143|3412|4321|| 2143|1234|4321|3412|| 3412|4321|1243|2134|| 4321|3412|2134|1243	**20736**	816	16920576

TABLE. ISOTOPY CLASSES OF TERNARY QUASIGROUPS OF ORDER 4

Proof. We use Corollary 1. There is only one isotopy class of unary quasigroups of order 4, and the unary quasigroup 1234 is the first unary quasigroup of 24 binary quasigroups. So, there are $24 \times 24 = 576$ binary quasigroups. There are 2 isotopy classes of binary quasigroups, C_1 with 144 and C_2 with 432 elements. The quasigroup $1234|2143|3412|4321 \in C_1$ is the first quasigroup of $b_1 = 132$ ternary quasigroups, and the quasigroup $1234|2143|3421|4312 \in C_2$ is the first quasigroup of $b_2 = 84$ ternary quasigroups. So, there are $144 \times 132 + 432 \times 84 = 55\,296$ ternary quasigroups of order 4. For the quaternary quasigroups of order 4 the results are presented in Table. □

We remark that the result of Table differs from that given in "On-Line Encyclopedia of Integer Sequences" (see [7]) for the sequence $A099321$ of "Number of isotopy classes of Latin cubes of order n". We note that, by using Table, we correctly computed the number of quaternary quasigroups of order 4 (see also [5, 6]).

Proposition 6. *The numbers of n-quasigroups of order* 5 *for* $n = 1$, $n = 2$ *and* $n = 3$ *are* 120, 161 280 *and* 2 781 803 520 *respectively.*

Proof. We use Corollary 1. There is only one isotopy class of unary quasigroups of order 5, and the unary quasigroup 12345 is the first unary quasigroup of 56 binary quasigroups of the form

$$12345|2a_1b_1c_1d_1|3a_2b_2c_2d_2|4a_3b_3c_3d_3|5a_4b_4c_4d_4.$$

So, 12345 can be the first unary quasigroup of $56 \times 4! = 1\,344$ binary quasigroups. Hence, there are $5! \times 1\,344 = 161\,280$ binary quasigroups of order 5.

There are 2 isotopy classes of binary quasigroups of order 5, C_1 with 17 280 and C_2 with 144 000 elements. The quasigroup 12345|31452|43521 |54213|25134 $\in C_1$ is the first quasigroup of $b_1 = 22\,584$ ternary quasigroups, and the quasigroup 12345|21453|34512|45231|53124 $\in C_2$ is the first quasigroup of $b_2 = 16\,608$ ternary quasigroups. So, there are $17\,280 \times 22\,584 + 144\,000 \times 16\,608 = 2\,781\,803\,520$ ternary quasigroups of order 5. □

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