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***Abstract*—In this paper we analyze synchronization in networks of identical and nearly identical Chua's oscillators. Using time-domain simulations and the Master Stability Function (MSF) approach in time and frequency domain, we show that losses can lower down the coupling bounds for which a given network of oscillators synchronizes. We also show that by using the extended MSF losses reduce the synchronization error when the oscillators are nearly identical, i.e. there is a bounded mismatch of the parameters of the oscillators.**

I. Introduction

Synchronization is exibited in many real systems and it has been an active research topic in the study of chaotic oscillators [1], [2]. A part of the research community has focused on synchronization in complex networks [3], [4], with a principal goal to determine the conditions under which the oscillators in the network synchronize or alter a given network to enhance its synchronizability [5]. Although, it is usually assumed that the constituent oscillators are ideal, in reality there are discrepancies between the parameters in the different oscillators [6], [7]. Hence, the Master Stability Function (MSF) approach proposed in [8] and widely used for networks of identical oscillators has been extended to nearly-identical oscillators, i.e. there is some bounded parameter mismatch, in [9], [10] and [11]. In another work [12] the authors experimentaly analyze synchronization in real networks of nearly-identical Chua circuits and confirm the theoretical findings of [11].

In this paper we further study the synchronization properties of coupled Chua's oscillators and show that the presence of a resistance R_0 (see Fig. 1), not considered in [12], can greatly reduce the boundary coupling strength from which synchronization is possible. Moreover, this resistance reduces the synchronization error in presence of parameter mismatch in networks of Chua's oscillators. As it is in reality, we consider that R_0 is present due to the loss of the inductor L in the Chua's oscillator. However, this resistance could come from an additional resistor added in series with the inductor.

The paper is organized as follows. In Section II, we recall the equations for the Chua's oscillator and the MSF approach for networks of Chua's oscillators. In Section III we analyze the stability of coupled identical Chua's oscillators in time and frequency domain, whereas in Section IV we analyze the synchronization error in time domain when the Chua's oscillators are nearly-identical. Some conclusions are drawn at the end of the paper.

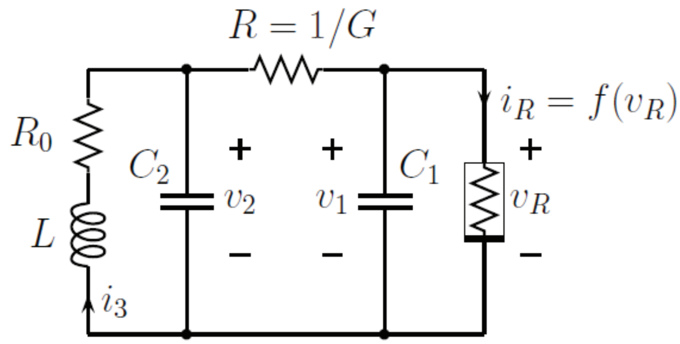

Figure 1. Chua's oscillator.

II. Preliminaries on Chua's oscillator and MSF

The dimensionless state equations of the Chua's oscillator [13] shown on Fig. 1 are:

$$\begin{aligned} \dot{x} &= \alpha[y - x - h(x)] \\ \dot{y} &= x - y + z \\ \dot{z} &= -\beta y - \gamma z \end{aligned} \quad (1)$$

where

$$h(x) = m_b x + 0.5(m_a - m_b)(|x+1| - |x-1|), \quad (2)$$

x,y and z are the state variables and $\alpha = C_2/C_1$, $\beta = C_2 R^2/L$, $\gamma = C_2 R R_0/L$, $m_a = RG_a$ and $m_b = RG_b$ are the parameters. We consider R_0 as a resistance due to the loss of the inductor, therefore, R_0 is always positive and its value depends on the quality factor Q of the inductor. Q is defined as $Q = w_0 L/R_0$, where w_0 is the internal frequency of the oscillator. In our simulations $w_0 = 18.48$ kHz and we consider $R_0 = 8.32\Omega$ ($Q = 40$) and $R_0 = 4.75\Omega$ ($Q = 70$), which are in the same range as the values given in [14]. Initially we analyze a network of N identical and symmetrically coupled Chua's oscillators represented with the following equations

$$\dot{x}_i = F(x_i) - g\sum_{j=1}^{N} L_{ij} H(x_j), \quad (3)$$

where $F(\cdot)$ is the dynamics of an isolated unit, $g > 0$ is a coupling constant, L is the network's Laplacian matrix, and $H(\cdot)$ is a coupling function.

In [8] and [5] the authors gave a significant contribution for examination of the (local) stability of the synchronous states ($x_1 = x_2 = \ldots = x_N = \bar{x}$), using the eigenvalues of the network's Laplacian matrix. The stability is determined by the variational equations and the motion on the synchronous manifold

$$\dot{\bar{x}} = F(\bar{x}). \tag{4}$$

The system is diagonalized in N blocks of a form given by Eq. (5), where y is a perturbation mode from the synchronized state; $DF(\bar{x})$ and $DH(\bar{x})$ denote the Jacobian matrices of F and H respectively, evaluated at $\bar{x}$; $\sigma = g\lambda_i$, $i = 1, \ldots, N$ are the coupling eigenvalues; $\lambda_1 = 0 < \lambda_2 \leq \ldots \leq \lambda_N$ are the eigenvalues of L, which are all real because L is symmetric.

$$\dot{y} = [DF(\bar{x}) - \sigma DH(\bar{x})]y \tag{5}$$

Having this, the master stability function (MSF) $\Lambda(\sigma)$ is defined as the maximum Lyapunov exponent of the system given by Eq. (4) and Eq. (5) as a function of σ [8]. The MSF indicates the stability of the synchronized state, which is unstable if $\Lambda(g\lambda_i) > 0$, for any $i \in \{2, \ldots, N\}$.

III. Stability Analyses of network of Chua's oscillators

Using the framework from Section II we show how the MSF changes as a function of the coupling strength in two different dynamical behaviors of the Chua's oscillator and different values of R_0. We focus on the following cases:

- **Case 1:** periodic Limit Cycle (LC) behavior with parameters $\alpha = 10.2$, $\beta = 22.22$, $m_a = -1.512$, $m_b = -0.818$, $\gamma = 0$ (i.e. the inductor is ideal and $R_0 = 0\Omega$);
- **Case 2:** chaotic Double Scroll (DS) behavior with parameters $\alpha = 10.2$, $\beta = 17.98$, $m_a = -1.36$, $m_b = -0.73$, $\gamma = 0$ ($R_0 = 0\Omega$).

In addition, for both cases we consider resistances $R_0 = 4.75\Omega$ ($\gamma = 0.0599$) and $R_0 = 8.32\Omega$ ($\gamma = 0.1049$).

A. Time domain stability analyses

From Fig. 2 (obtained from Eq. (4) and Eq. (5) in time domain) one can see that in Case 1 the presence of R_0 reduces the coupling strength needed to synchronize the network, i.e. when $R_0 = 0\Omega$, σ in the network should be approximately larger than 7.2, whereas when $R_0 = 8.32\Omega$, σ should be larger than 4.4 or in the interval $(0, 0.16]$ (see upper right corner of Fig. 2). However, when $\sigma \in (0, 0.16]$ and $R_0 = 8.32\Omega$, the oscillators are weakly synchronized. In addition, when $R_0 = 8.32\Omega$ and there is synchronization, there is a dumping factor, i.e. the values of the current and the voltages tend to an equilibrium point, as can be seen from the time evolution of the coupled state variables (see Fig. 3). When $R_0 = 4.75\Omega$ or lower there is no dumping factor of the signals in the oscillator from which we can conclude that $R_0 = 4.75\Omega$ is a critical value of the resistance for appearance of a dumping factor. When $R_0 = 4.75\Omega$, σ should be approximately bigger than 5 and there is no synchronization for very small couplings as with $R_0 = 8.32\Omega$. From Fig. 4 one can see that in Case 2,

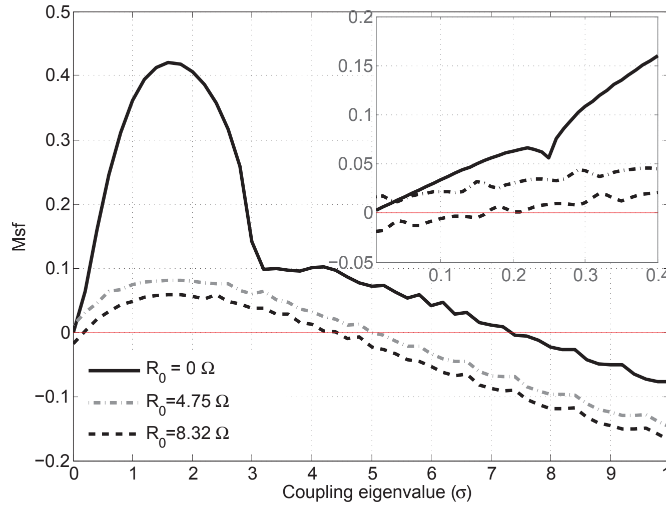

Figure 2. MSF for Case 1 for different coupling eigenvalues and different values for R_0.

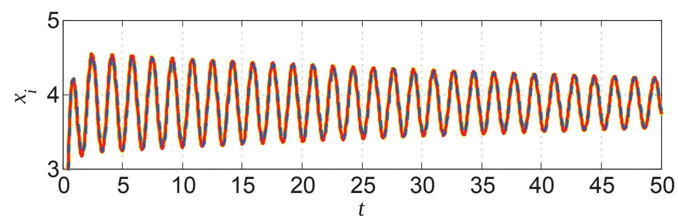

Figure 3. Time evolution of the coupled state variables x_i, $i = 1, 2, 3, 4$ for Case 1 when $R_0 = 8.32\Omega$.

in presence of R_0, the network can synchronize for smaller values of σ, i.e. when $R_0 = 0\Omega$, σ should be approximately larger than 6.2, whereas when $R_0 = 8.32\Omega$, σ should be approximately larger than 4.8. In addition, when $R_0 = 4.75\Omega$, σ should be approximately larger than 5.2. With Fig. 2 and Fig. 4 we show that R_0 reduces the coupling boundary at which the synchronization in the network can happen. For instance, in Case 1 when $R_0 = 8.32\Omega$ the boundary is reduced by approximately 40% and in Case 2 by 23%. However, in Case 1 this value of R_0 introduces a dumping factor, but it also introduces another range $(0, 0.16]$ in which weak synchronization is possible. Instead, when $R_0 = 4.75\Omega$ in Case 1 the boundary is approximately reduced by 31.5%, while in Case 2 by 16%.

B. Frequency domain stability analyses

As a confirmation of the results from Subsection III-A here we show the results for Case 1 when using the MSF approach in the frequency domain [15], [16]. This approach, as presented in [15], determines the MSF $\Lambda(\sigma)$ for σ in $[\sigma_*, \sigma^*]$ on a limit cycle in four steps: (1) for a given oscillator $\dot{x} = f(x)$ finds a good approximation of the regarded limit cycle X^F; (2) solves $(\Gamma_D B_M \Gamma_D^{-1} - \omega\Omega_M)U^F = \mu_d U$ for σ in $[\sigma_*, \sigma^*]$ with B_M built using B given by $Df + \sigma H$ evaluated on X^F, thus obtaining $D \cdot M$ eigenvalues; (3) selects

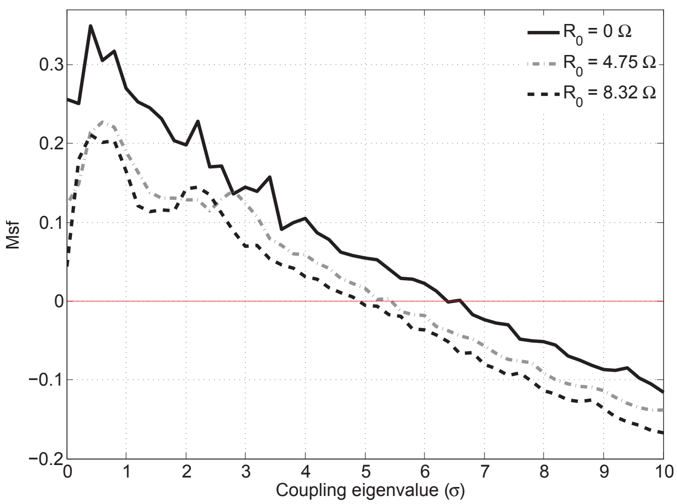

Figure 4. MSF for Case 2 for different coupling eigenvalues and different values for R_0.

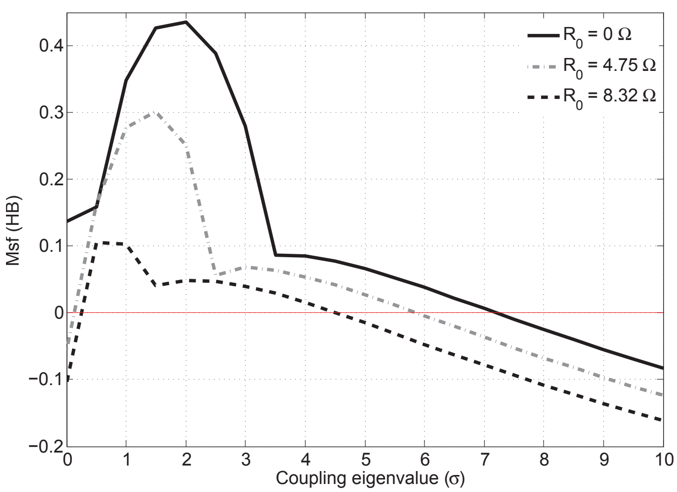

Figure 5. MSF in frequency domain for Case 1 for different coupling eigenvalue and different values for R_0.

the D eigenvalues with smallest imaginary part $\mu_1, ..., \mu_D$; (4) $\Lambda(\sigma) = max\{\Re(\mu_1), ..., \Re(\mu_D)\}$, where Γ_D is a block-diagonal matrix consisted of D copies of Γ (matrix used to link the Fourier coefficients and the time samples), B_M is a matrix created by expanding the elements of B in a diagonal block of time samples $B(t_1), ...B(t_M)$, and Ω_M is a block diagonal matrix consisted of D copies of Ω, $B(t)$ is the Jacobian matrix of the vector field f on a limit cycle solution, D_f is the Jacobian of f, μ_d and U are the solution of the corresponding eigenvalue problem.

Using this method we can derive the same conclusion as in Subsection III-A, i.e. when $R_0 = 8.32\Omega$ and $R_0 = 4.75\Omega$ the network of oscillators synchronize for approximately the same range (compare Fig. 2 and Fig. 5).

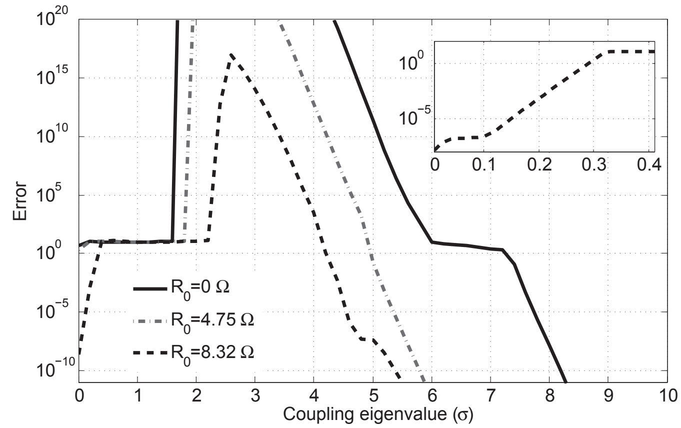

Figure 6. Time domain simulations of the synchronization error for the network of 4 Chua's oscillator (Case 1) for different coupling eigenvalues and different values for R_0.

C. Critical discussion

The time domain simulations for a network of four fully connected Chua's oscillators justify our conclusion that the resistance R_0 is of great importance for the coupling strength needed to get full synchronization. From Fig. 6 (upper right corner) we can conclude that for Case 1 when $R_0 = 8.32\Omega$ and for low coupling eigenvalue (i.e. $\sigma \leq 0.16$) the synchronization error of the network is around 10^{-5}, which means that the network weakly synchronizes, while this is not the case when R_0 is close to 0. In addition, for $R_0 = 8.32\Omega$ the network synchronizes when $\sigma \geq 4.8$, whereas when $R_0 = 0\Omega$ the coupling eigenvalue should be larger than 7.7. On the other side, for $R_0 = 4.75\Omega$ there is no synchronization for small σ, however, the networks synchronize for σ larger than 5.2. In Case 2 the right boundary coupling eigenvalue (σ) was approximately reduced by 20% for $R_0 = 8.32\Omega$.

As a remark, the results in this section are obtained using coupling just on the variable x, i.e. $h(w) = [x, 0, 0]'$, when we have coupling on all variables, i.e. $h(w) = [x, y, z]'$ the results (not shown) with R_0 and without R_0 do not differ too much, i.e. the difference between the coupling strength needed for synchronization is much smaller.

IV. Analyses of the synchronization error of nearly identical coupled Chua's oscillators

The MSF can be used for stability analysis only when the parameters in the system are ideal, however, in reality, oscillators with exactly the same parameters are impossible and usually the elements in the oscillator have some nominal tolerance lower than 5%.

In [11] authors give definition of the *extended master stability equation* for coupled nearly-identical systems:

$$\dot{y} = [D_w F(\bar{x}) - \sigma \cdot DH(\bar{x})]y + D_\mu F(\bar{x}) \cdot \psi \qquad (6)$$

where they introduced another parameter $\psi \in R^p$. After determing the stability of Eq. (6) depending on σ and ψ, the i-th eigenmode stability is found by assigning $\sigma = g\lambda_i$ and

$\psi = U \cdot \delta\mu$, where $\delta\mu$ is the parameter mismatch vector and the columns of the matrix U are the left eigenvectors of L. The asymptotic value of the y's norm is assigned to be an *extended master stability function* $\Omega(\sigma, \psi)$ (eMSF), as the homogeneous part of the solution is asymptotically stable for σ (see Eq. (5)). For networks with symmetrical coupling, $\Omega(\sigma, \psi)$ provides estimation of the square-sum synchronization error:

$$\sum_{i=1}^{N} ||\eta_i(t)||^2 \xrightarrow{t\to\infty} \sum_{i=2}^{N} \Omega(\sigma_i, \psi_i)^2 \tag{7}$$

where $||\cdot||$ indicates an Euclidean norm.

A validation of this eMSF approach was given in [12], where the authors using numerical simulations and experiments confirm that the eMSF gives an accurate estimate of the boundary coupling which guaranties synchronization in network of non-identical Chua circuits.

Using the nominal tolerance of 1% for the resistors (R_1, R_0), 5% for the conductors (C_1, C_2) and the inductance L, we have calculated the worst case value of the nominal tolerance for the dimensionless parameters. Thus, when changing the parameter mismatch ψ from 0 to 0.1 (i.e. from 0% to 10%), actually we are changing α from 0% to 10%, β and γ from 0% to 12%, and m_a and m_a from 0% to 1%.

In the following, we investigate the effect that the resistance R_0 has on the synchronization error when the oscillators are nearly identical. We have calculated that the resistance R_0 introduces significant error reduction. Moreover, the error is reduced as the coupling strength grows for both Case 1 and 2 (results are not shown).

On Fig. 7 the synchronization error as predicted by Eq. (7) is compared to the actual error for Case 1. The synchronization error grows as the parameter mismatch rises and the actual error is well predicted by Eq. (7) for mismatch of up to $\Psi = 0.05$. For higher values of R_0, the error is lower.

Similar results were obtained for Case 2. However, in this case the reduction of the error was lower and the actual error was well predicted for mismatch of up to $\Psi = 0.15$.

V. Conclusion

In this paper we have analyzed the synchronization in networks of coupled identical and nearly identical Chua's oscillators. We have shown that the presence of R_0 reduces the value of the coupling strength needed for synchronization and it also creates small range of small couplings for which weak synchronization is possible. Furthermore, using the extended Master Stability Function, we have shown that this resistance lowers down the error of synchronization when the oscillators are nearly identical, particularly for limit cycle behavior. Comparing the results of the eMSF approximation of the synchronization error with the actual synchronization error, it can be noticed that the eMSF approach is a good approximation when the parameter mismatch is not bigger than 5% of the nominal values, which means that in reality it gives a good approximation of the synchronization error.

As a final remark, large values of R_0 can introduce a dumping factor on the current and the voltages in the Chua's oscillator and we have found a threshold value of R_0 for which this factor is not present.

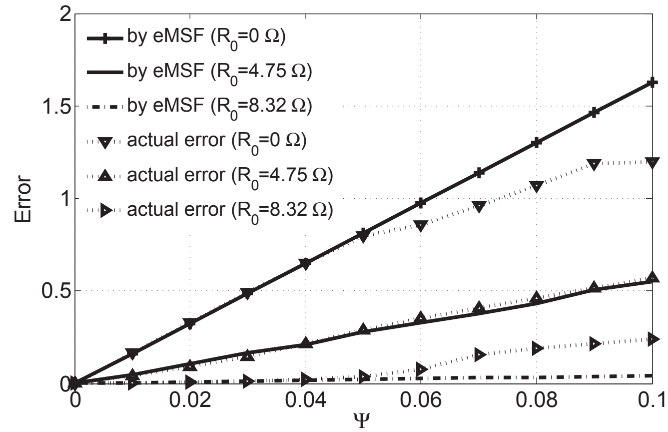

Figure 7. Comparison of the synchronization error predicted by eMSF with the actual error for a network of 4 Chua's oscillators (Case 1) for 3 different values of R_0.

Acknowledgments

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