Deterministic Diffusion in a Gravitational Billiard

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Numerical study of the motion of a classical particle in a homogeneous gravitational field bouncing off elastically from a piecewise linear periodic boundary shows that it is characterized by diffusion constant, but nonlinear scaling of the mean-square displacement is also observed. It is shown that periodic by modulus trajectories cannot have a segment with vertical flight.

§1. Introduction

The phenomenon of random looking behaviour in deterministic systems is subject of widespread study in many branches of modern physics. This is also the case in the field of transport properties such as diffusion. Classically diffusion is viewed as a stochastic process caused by random forces acting on particles and the process is modelled with Langevin equation. Recently a different point of view has been established where strictly deterministic systems have been shown to display diffusion. Most frequently such systems are represented with one-dimensional maps, usually taking piecewise linear maps.[1),2)] Another popular model is the standard map.[3)] Our motivation was to construct a physical model which has diffusive properties and can be solved to a large extent analytically without the need to introduce any kind of approximation in the calculations.

§2. Definition of the model and its analysis

We consider classical particles moving in a homogeneous gravitational field. Their motion is restricted from below with a periodic piecewise linear boundary. The collisions of the particle with the boundary are elastic and the particle continues its motion with the same energy after the reflection, with reflection angle equal to the incident angle as is the case in geometric optic. This billiard system was considered by Lehtihet and Miller[4)] who restricted the motion to a single infinite wedge shaped well. Physical realization of such system was achieved recently by two groups of researchers[5)] experimenting with ultracold atoms bouncing off beams of light. Lorenz has written about another more prosaic realization of a similar system in the form of a pinball machine.[6)] In a constant gravitational field classical particles follow parabolic paths. The intersection point of the trajectory with the boundary is a solution of quadratic equation. Applying the reflection condition one obtains initial conditions for the next parabolic flight of the particle.

The energy of a point particle of mass m moving under constant vertical gravitational acceleration g is

$$E = \frac{p_x^2 + p_y^2}{2m} + mgy \ ,$$

where x and y denote horizontal and vertical coordinates, while p_x and p_y represent the corresponding momenta. It is possible to set $m = 1$ and $g = 1$ which is equivalent to the use of some special time unit. We will be interested in motion of particles with sufficient energy E to overcome the barrier $H = l \tan\theta \equiv kl$ between the wells.

As an illustration of the typical situation, in Fig. 1(a) the Poincaré surface of section with the plane $p_y = 0$ is shown for $E = 3$ (expressed in units of $E_0 = mgH = H$) and $k = 1.2$. The horizontal and vertical direction correspond to x and y coordinates of the particle, the coordinate x being taken by modulus $2l$. All the intersecting points are generated by a single chaotic trajectory. The empty elliptic areas in Fig. 1(a) are reserved for quasiperiodic trajectories which are organized around central periodic or periodic by modulus trajectories. Such trajectories corresponding to some of the largest islands in Fig. 1(a) are depicted in Fig. 2. Three of them are periodic (Figs. 2(a), (b), (c)) and one is periodic by modulus (Fig. 2(d)). Two of the periodic orbits (Figs. 2(a) and (b)) at two stages are reflected back orthogonally after collision with billiard boundary, while the remaining periodic trajectory (Fig. 2(c)) has two 'ends' with vertical flights which return backward after reaching the maxima.

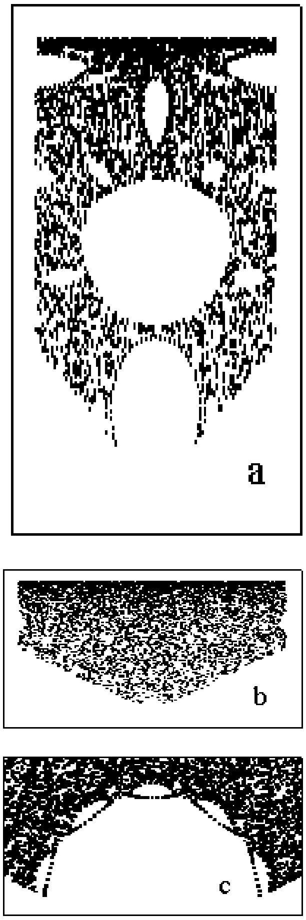

Fig. 1. Poincaré surface of section $p_y = 0$: (a) $E = 3$, $k = 1.2$; (c) lowest part of Fig. 1(b) under magnification.

The variety of all possible trajectories can be classified as follows: 1) periodic and periodic by modulus, 2) quasiperiodic, and 3) chaotic. The quasiperiodic trajectories surrounding a periodic trajectory are confined to a finite interval along the x-axis, while the quasiperiodic trajectories surrounding the periodic by modulus trajectories are unbound. Similarly, chaotic trajectories can be bound or unbound.

The diffusion coefficient is defined by Einstein's relation

$$D = \lim_{t\to\infty} \frac{1}{2t} \langle [x(t) - x(0)]^2 \rangle \ ,$$

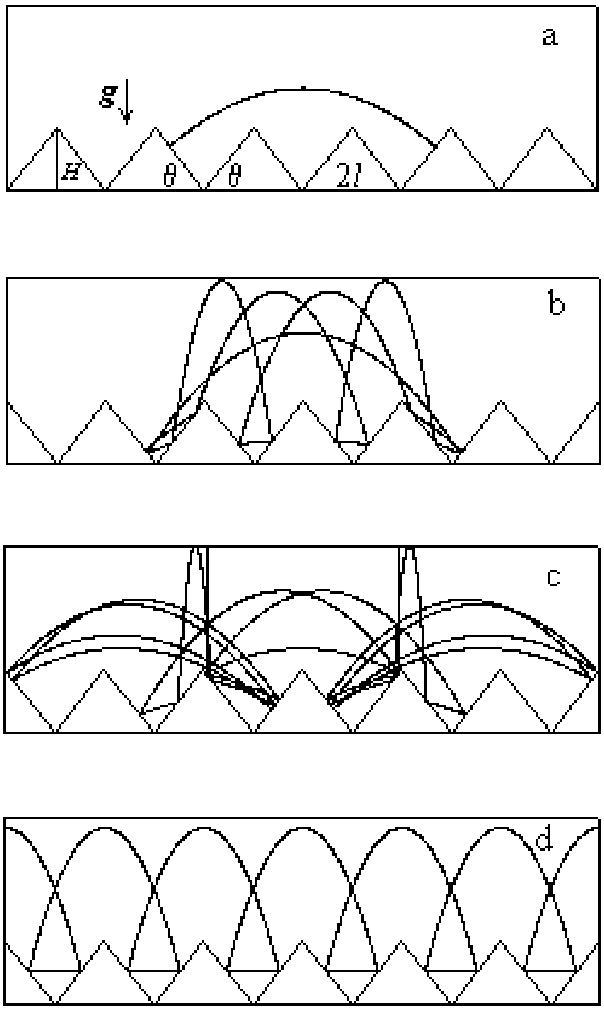

Fig. 2. Periodic trajectories with orthogonal reflection from the boundary (a),(b), periodic trajectory with vertical flightand periodic by modulus trajectory (d) for $E = 3$, $k = 1.2$. Some of the parameters of the system are defined in (a).

where the average $\langle \cdots \rangle$ is taken over a statistical ensemble of particles with uniformly distributed initial conditions. Contribution to the diffusion comes from the unbound orbits. From the definition it becomes evident that quasiperiodic by modulus trajectories make dominant contribution to the mean-square displacement $\langle [x(t) - x(0)]^2 \rangle$, which scales as t^2 if such trajectories are not excluded and makes meaningless the definition of the diffusion coefficient. It is therefore desirable to separate out quasiperiodic by modulus motion and this is possible by making the following observation.

Consider the set of initial conditions with arbitrary x and y coordinates, $p_y = 0$ and p_x such that the energy has a constant value. All possible trajectories within given energy surface are included in this set because each trajectory at some stage of

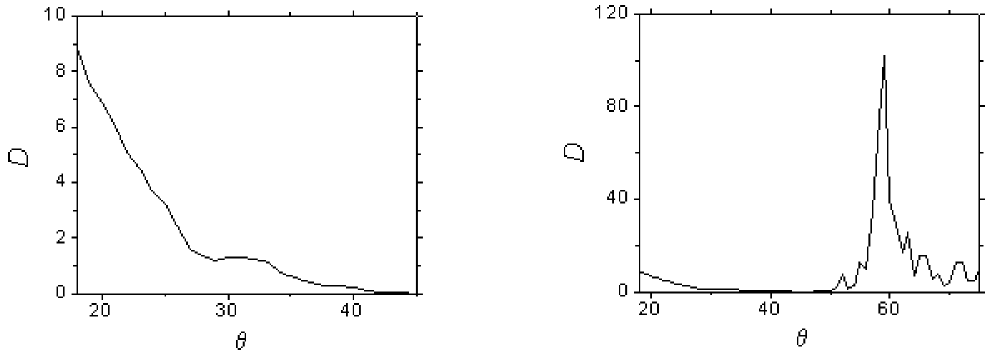

Fig. 3. Diffusion coefficient D as a function of the angle θ for $E = 3$.

the motion reaches its maximum defined by the condition $p_y = 0$. On the other hand not all trajectories have sections with vertical flight which at such stage satisfy the condition $p_x = 0$. The set of trajectories with initial conditions given by arbitrary x and y coordinates, $p_x = 0$ and p_y determined from the constant energy condition includes periodic orbits but does not include trajectories periodic by modulus. This can be seen by assuming that there is a periodic by modulus trajectory with initial condition $p_x = 0$. From the reversibility of the classical equations of motion, it follows that the same trajectory should be also periodic. This incompatibility proves that periodic by modulus trajectories are excluded by restricting the initial conditions to vertical falls only. Since quasiperiodic by modulus trajectories are located in the immediate vicinity of those which are periodic by modulus, by considering only trajectories with initial vertical fall, we could eliminate the contribution to the mean-square displacement of trajectories which are quasiperiodic by modulus. The validity of the above argument is applicable to any periodic billiard boundary and not only to the particular one which is under consideration.

To obtain the diffusion coefficient we have followed the trajectories of 10000 particles and the results are shown graphically in Fig. 3 as a function of the angle θ. For $\theta < 45°$ the calculations were done separately for two different sets of initial conditions, with a restriction $p_x = 0$, and $p_y = 0$, respectively. The difference between the results appears insignificant. Smaller angles $\theta < 18°$ are not included because the necessary time length to achieve limiting behaviour is beyond our computational means.

When the angle θ is very small it is observed that for some intermediate time interval the mean-square displacement scales as t^α with $\alpha > 2$. In particular for $k = 0.005$, $\alpha = 2.6$ was obtained. For very small θ, significant part of any trajectory is limited to bounces from one of the boundaries within a single well and therefore represents motion which is accelerated for long time along the horizontal direction. We expect that in the limit $t \to \infty$, normal diffusion is recovered.

As a further evidence that the system of particles is approaching Gaussian distribution we have calculated some higher order moments. For a Gaussian random variable ξ with vanishing mean value, one has $k_4 = \langle\xi^4\rangle/\langle\xi^2\rangle^2 = 3$ and $k_6 = \langle\xi^6\rangle/\langle\xi^4\rangle\langle\xi^2\rangle = 5$. Our simulations with 1000 particles typically differ from the theoretical results by less than 10%.

§3. Conclusions

We have studied simple but realistic physical system which is fully deterministic and shows clear diffusive properties. A simple way was found how to exclude the contribution to the mean-square displacement of trajectories flying in one direction without interruption. Further studies of the model are possible. For example, preliminary investigations have shown that the presence of noise in the form of quenched disorder of the angle θ in various wells generally leads to increased diffusion.

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