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Monoids of Powers in Varieties of f-Idempotent Groupoids

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Abstract

The monoids of powers in varieties of f-idempotent groupoids and commutative f-idempotent groupoids, where f is an irreducible groupoid power with a length at least 3, are constructed. It is shown that these monoids are free over a countable set of irreducible groupoid powers.

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1 Introduction and preliminary notes

A groupoid $\boldsymbol{G} = (G, \cdot)$ is an algebra with one binary operation. We denote by $\mathbf{T}_X = (T_X, \cdot)$ the groupoid of terms over a nonempty set X. The terms are denoted by $t, u, v, w, \ldots$ For any term v of T_X, the *length* $|\,v\,|$ of v and the *set of subterms* $P(v)$ of v are defined as:

$$|\,x\,| = 1,\ |\,tu\,| = |\,t\,| + |\,u\,|;\ P(x) = \{x\},\ P(tu) = \{tu\} \cup P(t) \cup P(u),$$

for any $x \in X$ and any $t, u \in T_X$. We denote by $\mathbf{E} = (E, \cdot)$ the groupoid of terms over one-element set $\{e\}$. The elements of E are called *groupoid powers* and are denoted by $f, g, h, \ldots$. They are introduced in [6] and are used in several papers: [1], [2], [7].

Further on we use the shortlex ordering of terms (denoted by $\leq$), i.e. terms are ordered so that a term of a particular length comes before any longer term, and amongst terms of equal length lexicographical ordering is used, with terms earlier in the lexicographical ordering coming first. Then T_X is a linearly ordered set.

A term t is said to be *order-regular* if $t \in X$ or ($t = t_1t_2$ and $t_1 \leq t_2$). Specially, a groupoid power f is said to be *order-regular* if $f = e$ or ($f = f_1f_2$ and $f_1 \leq f_2$).

We denote by $\mathbf{T}_c = (T_c, \odot)$ the free commutative groupoid over X defined in [2] by: $T_c = \{t \in T_X :$ every subterm of t is order-regular$\}$ and the operation $\odot$ by $t, u \in T_c \Rightarrow (t \odot u = tu$ if $t \leq u; t \odot u = ut$ if $u < t)$. We denote by $\mathbf{E}_c = (E_c, \odot)$ the free commutative groupoid over $\{e\}$.

For any groupoid $\boldsymbol{G} = (G, \cdot)$, each groupoid power f induces a transformation $f^{\boldsymbol{G}} : G \to G$, called the *interpretation* of f in $\boldsymbol{G}$, defined by: $f^{\boldsymbol{G}}(a) = \varphi_a(f)$ for every $a \in G$, where $\varphi_a : E \to G$ is the homomorphism from $\mathbf{E}$ into $\boldsymbol{G}$ such that $\varphi_a(e) = a$. In other words, for any $f_1, f_2 \in E$ and $a \in G$,

$$e^{\boldsymbol{G}}(a) = a, \ \ (f_1f_2)^{\boldsymbol{G}}(a) = f_1^{\boldsymbol{G}}(a)f_2^{\boldsymbol{G}}(a).$$

We write $f(a)$ instead of $f^{\boldsymbol{G}}(a)$ when $\boldsymbol{G}$ is understood, and specially when $\boldsymbol{G} = \mathbf{T}_X$ or $\boldsymbol{G} = \mathbf{E}$.

Define an other operation "$\circ$" on E by $f \circ g = f(g)$. Clearly, the interpretation of f in $\mathbf{E}$ is presented as:

$$e \circ g = g \circ e = g, \ \ (f_1f_2) \circ g = (f_1 \circ g)(f_2 \circ g), \tag{1}$$

for any $g, f_1, f_2 \in E$. Note that, for any groupoid $\boldsymbol{G}$, the triple $(E^{\boldsymbol{G}}, \circ', e^{\boldsymbol{G}})$, where $E^{\boldsymbol{G}} = \{f^{\boldsymbol{G}} : f \in E\}$ and $\circ'$ is an operation in $E^{\boldsymbol{G}}$ defined by $f^{\boldsymbol{G}} \circ' h^{\boldsymbol{G}} = (f \circ h)^{\boldsymbol{G}}$, is a monoid. The following statements are shown in [6].

Proposition 1.1 *If $f, g \in E$ and $t, u \in T_X$, then:*
a) $|\,f(t)\,| = |\,f\,| \cdot |\,t\,|$; $t \in P(f(t))$;
b) $f(t) = f(u) \Rightarrow t = u$;
c) $f(t) = g(u) \wedge (|\,t\,| = |\,u\,| \ \vee \ |\,f\,| = |\,g\,|) \ \Leftrightarrow \ (f = g \wedge t = u)$;
d) $f(t) = g(u) \wedge |\,t\,| > |\,u\,| \ \Leftrightarrow (\exists !\, h \in E) \ (t = h(u) \ \wedge \ g = f(h))$.

The corresponding translation of Prop.1.1 when T_X is substituted by E is obvious.

A groupoid power f is said to be *irreducible* in $(E, \circ, e)$ if $f \neq e$ and $f = g \circ h \Rightarrow g = e \ \vee \ h = e$. It is *reducible* in $(E, \circ, e)$ if $f = g \circ h$ for some $g, h \in E \setminus \{e\}$. If $f = g \circ h$, then g and h are *left* and *right divisors* of f, respectively.

It is shown in [6] that $(E, \circ, e)$ *is a free cancelative monoid over the set of irreducible groupoid powers* and that *the monoids* $(E, \circ, e)$ *and* $(E_c, \circ, e)$ *are isomorphic.*

2 Examples

Let $\mathcal{V}$ be a variety of groupoids and let $\mathbf{E}_{\mathcal{V}} = (E_{\mathcal{V}}, \cdot)$ be the free groupoid in $\mathcal{V}$ over $\{e\}$. The elements of $E_{\mathcal{V}}$ can be considered as powers in groupoids of $\mathcal{V}$. Namely, for every $\boldsymbol{G} \in \mathcal{V}$ and $f \in E_{\mathcal{V}}$, we define a transformation $f^{\boldsymbol{G}}$ as an interpretation of f in $\boldsymbol{G}$. We say that $f^{\boldsymbol{G}}$ is a $\mathcal{V}$*-power* in $\boldsymbol{G}$ ([6]). We denote by $\mathcal{V}_c$ the variety of commutative groupoids in $\mathcal{V}$ and by $(E_{\mathcal{V}}, \circ, e)$ and $(E_{\mathcal{V}_c}, \circ, e)$ the monoids of powers in $\mathcal{V}$ and $\mathcal{V}_c$, respectively. For any variety $\mathcal{V}$ of groupoids the following two questions arise: (A) Is the monoid $(E_{\mathcal{V}}, \circ, e)$ free? (B) Are the monoids $(E_{\mathcal{V}}, \circ, e)$ and $(E_{\mathcal{V}_c}, \circ, e)$ isomorphic?

The following examples show that, in general, both questions may have negative answers. We use the free groupoids constructed in [5], [4] and [3].
Example 2.1 Let $\mathcal{V} = \mathtt{Var}(x^2 \approx x)$. The groupoid $\boldsymbol{R}_{\mathcal{V}} = (R_{\mathcal{V}}, *)$ defined by $R_{\mathcal{V}} = \{t \in T_X : (\forall u \in T_X)\ u^2 \notin P(t)\}$ and

$$t, u \in R_{\mathcal{V}} \Rightarrow [t * u = tu, \text{if } u \neq t;\ t * u = t, \text{if } u = t],$$

is $\mathcal{V}$-free over X. The carrier of the free groupoid $\mathbf{E}_{\mathcal{V}} = (E_{\mathcal{V}}, *)$ over $\{e\}$ is $E_{\mathcal{V}} = \{e\}$, i.e. the $\mathcal{V}$-powers are trivial. The monoid $(E_{\mathcal{V}}, \circ, e)$ is not free. The monoid $(E_{\mathcal{V}_c}, \circ, e)$ coincides with $(E_{\mathcal{V}}, \circ, e)$ and thus these two monoids are isomorphic. Hence, the answer of (A) is negative and of (B) is positive for this variety.
Example 2.2 Let $\mathcal{U} = \mathtt{Var}(x^2y^2 \approx xy)$. The conjunction of the identities $xy^2 \approx xy$ and $x^2y \approx xy$ is an axiom system of $\mathcal{U}$ as well ([4]). The carrier of the free groupoid $\mathbf{E}_{\mathcal{U}} = (E_{\mathcal{U}}, *)$ over $\{e\}$ is $E_{\mathcal{U}} = \{e, e^2\}$ where $e * e = e^2 = e * e^2 = e^2 * e = e^2 * e^2$. The monoid $(E_{\mathcal{U}}, \circ, e)$ is not free. The monoid $(E_{\mathcal{U}_c}, \circ, e)$ coincides with $(E_{\mathcal{U}}, \circ, e)$ and thus these two monoids are isomorphic. Hence, the answer of (A) is negative and of (B) is positive for the variety $\mathcal{U}$.
Example 2.3 Let $\mathcal{W} = \mathtt{Var}(xy^2 \approx xy)$. Then the following identities hold in $\mathcal{W}$ for any $m, n \in \mathbf{N}$: $xy^n \approx xy$ and, specially, $x^n x^m \approx x^{n+1}$, where x^k is defined by $x^1 = x$, $x^{k+1} = x^k x$. As a special case of the main result of [4], we obtain that the groupoid $(E_{\mathcal{W}}, *)$, defined by $E_{\mathcal{W}} = \{e^n : n \in \mathbf{N}\}$ and $e^n * e^m = e^{n+1}$ is free in $\mathcal{W}$ over $\{e\}$. It remains to define a monoid $(E_{\mathcal{W}}, \circ, e)$ that satisfies the conditions (1).

Assume that $(E_{\mathcal{W}}, \oplus, e)$ and $(E_{\mathcal{W}}, \otimes, e)$ are two monoids that satisfy (1). By induction on length $|\,f\,|$ one can show that for all $f, g \in E_{\mathcal{W}}$, $f \oplus g = f \otimes g$, i.e. there is at most one groupoid with the desired properties. The conditions (1) suggest the following definition of $\circ$: $e^m \circ e^n = e^{m+n-1}$. One can show that $(E_{\mathcal{W}}, \circ, e)$ is a free monoid over $\{e^2\}$ (e^2 is the only irreducible power), i.e. $(E_{\mathcal{W}}, \circ, e)$ is isomorphic with the additive monoid of nonnegative integers $(\mathbf{N}_0, +, 0)$. By Example 2.2 we obtain that $\mathcal{W}_c = \mathcal{U}$ and, moreover, $E_{\mathcal{U}} = E_{\mathcal{U}_c} = E_{\mathcal{W}_c} = \{e, e^2\}$ and $(E_{\mathcal{U}}, \circ, e) = (E_{\mathcal{U}_c}, \circ, e) = (E_{\mathcal{W}_c}, \circ, e)$, where "$\circ$" is defined by: $e \circ e = e, e \circ e^2 = e^2 \circ e = e^2 \circ e^2 = e^2$. Thus, the monoids $(E_{\mathcal{W}}, \circ, e)$ and $(E_{\mathcal{W}_c}, \circ, e)$ are not isomorphic. Therefore, the answer of (A) is positive and of (B) is negative in this case.

3 Powers in f-idempotent groupoids

Let f be a fixed element of $E \setminus \{e\}$ and denote by $\mathcal{V}^f$ the variety of groupoids satisfying the identity $f(x) \approx x$. The elements of $\mathcal{V}^f$ are called *f-idempotent groupoids.* For $f = e^n$, the identity $f(x) \approx x$ becomes $x^n \approx x$, where x^k is defined as in Example 2.3, and the groupoids are said to be *n-idempotent.* This variety is denoted by $\mathcal{V}^{(n)}$.

It is shown in [6] that *the monoid of groupoid powers in the variety $\mathcal{V}^{(n)}$, $n \geq 3$, is free over the set of irreducible powers that belong to the set* $E_{\mathcal{V}^{(n)}} = \{g \in E : (\forall h \in E)\ \ h^n \notin P(g)\}$. Is an analogous result true for the variety $\mathcal{V}^f$, if f is any groupoid power? We will ask for an answer to this question when f is any irreducible element (not necessarily $f = e^n$).

Free groupoids in the variety $\mathcal{V}^f$ when f is an irreducible element in $(E, \circ, e)$ are described in [5] (Th.1, Th.2). Namely, R_f ($\subseteq T_X$) is defined by

$$R_f = \{u \in T_X : (\forall t \in T_X)\ \ f(t) \notin P(u)\}$$

and an operation $*$ on R_f is defined by:

$$u, v \in R_f \Rightarrow u * v = \begin{cases} uv, & \text{if } uv \in R_f \\ t, & \text{if } uv = f(t). \end{cases}$$

Then $\boldsymbol{R}_f = (R_f, *)$ is a free groupoid in $\mathcal{V}^f$ over X. Put

$$E_f = \{g \in E : (\forall h \in E)\ \ f(h) \notin P(g)\}.$$

Here E_f stands for $E_{\mathcal{V}^f}$. Clearly, $f \neq e$, since f is irreducible in $(E, \circ, e)$. We will show that for any irreducible $f \in E$, $|\,f\,| \geq 3$, $(E_f, \circ, e)$ is a free monoid over a countable generating set.

Proposition 3.1 *$(E_f, \circ, e)$ is a submonoid of $(E, \circ, e)$.*

Proof. It suffices to show that $g \circ h \in E_f$, for any $g, h \in E_f$. Suppose that there are $g, h \in E_f$ such that $g \circ h \notin E_f$. Clearly, $g \neq e$. Let $g \in E_f$ be an element with the smallest length, such that $g \circ h \notin E_f$. Then, for $g = g_1 g_2$, by induction on $|\,g\,|$, we have $g_1 \circ h$, $g_2 \circ h \in E_f$. Since $g \circ h \notin E_f$, it follows that there is $p \in E$ such that $g \circ h = f \circ p$. The following three cases are possible: $|\,h\,| = |\,p\,|$, $|\,h\,| > |\,p\,|$ and $|\,h\,| < |\,p\,|$. We obtain a contradiction in all the cases using Prop.1.1, namely b) for the first case and c) for the last two cases. Hence, $g \circ h \in E_f$.

Proposition 3.2 *If $g \circ h \in E_f$, then $g, h \in E_f$.*

Proof. If $g = e$ or $h = e$, then the claim is obvious. Assume that $g \neq e \neq h$. Since $g \circ h \in E_f$, it follows that for every $p \in E$, $f \circ p \notin P(g \circ h)$. By induction on $|\, g \,|$ one can show that $h \in P(g \circ h)$ and thus $P(h) \subseteq P(g \circ h)$. Therefore, $f \circ p \notin P(h)$ which implies that $h \in E_f$. Suppose that there are $g, h \in E$ such that $g \circ h, h \in E_f$ and $g \notin E_f$. Since $g \notin E_f$, it follows that $g = f \circ p$, for some $p \in E$, and thus we would have $g \circ h = (f \circ p) \circ h = f \circ (p \circ h)$. This is a contradiction, since $g \circ h \in E_f$ and $f \circ (p \circ h) \notin E_f$.

Proposition 3.3 *An element $g \in E_f$ is irreducible in $(E_f, \circ, e)$ if and only if g is irreducible in $(E, \circ, e)$.*

Proof. Suppose that g is reducible in $(E, \circ, e)$. Then $g = p \circ q$ for some $p, q \in E \setminus \{e\}$. By Prop.3.2, $p \circ q \in E_f$ implies that $p, q \in E_f$ and thus g is reducible in $(E_f, \circ, e)$, that contradicts the assumption. The converse is clear.

It is shown in [6] that *for every $p \in E \setminus \{e\}$ there is a unique sequence $p_1, \ldots, p_n$ of irreducible elements in $(E, \circ, e)$ such that $p = p_1 \circ p_2 \circ \ldots \circ p_n$.* As a consequence of this and Prop.3.3 we obtain:

Corollary 3.4 *For every $h \in E_f \setminus \{e\}$, there is a uniquely determined sequence $q_1, q_2, \ldots, q_n$ of irreducible elements in E_f such that $h = q_1 \circ q_2 \circ \ldots \circ q_n$.*

Proposition 3.5 *There are countably many irreducible elements in the monoid $(E_f, \circ, e)$, when $|\, f \,| \geq 3$.*

Proof. If $|\, f \,| = 3$, then $f = e^2 e$ or $f = ee^2$. They are both irreducible in $(E, \circ, e)$. Take $f = e^2 e$. Put $q_1 = e^2, q_{k+1} = eq_k$. For every positive integer k there is an irreducible element $q_k \in E_f$. Thus, there are infinitely many irreducible elements in E_f. Symmetrically for $f = ee^2$.

Let $f = e^n$, for a fixed $n \geq 3$. (Note that e^n is an irreducible element in $(E, \circ, e)$, for every positive integer $n \geq 2$.) Then $q_1 = \underbrace{e(\ldots(e(ee))}_{n-1}$, $q_{k+1} = eq_k$ defines an infinite sequence of irreducible elements in $(E_f, \circ, e)$. In general, let f, $|\, f \,| = n \geq 3$, be irreducible in $(E, \circ, e)$. Then $e^2 e \in P(f)$ or $ee^2 \in P(f)$. If $e^2 e \in P(f)$, then we define q_1 to have the same "form" as f, putting $e^2 e$ instead of ee^2. Then: $q_1 \in E_f$ is irreducible in $(E_f, \circ, e)$ and the sequence defined by: q_1, $q_{k+1} = eq_k$ consists of irreducible elements in $(E_f, \circ, e)$. Similarly for the case $ee^2 \in P(f)$.

By Prop. 3.1, 3.5 and Cor.3.4 it follows:

Theorem 3.6 *For every irreducible element $f \in E$, $|\, f \,| \geq 3$, the monoid $(E_f, \circ, e)$ is a free monoid over a countable set of irreducible elements in $(E, \circ, e)$.*

4 The commutative case

Let $\mathcal{V}^{fc}$ be the variety of commutative f-idempotent groupoids. First, we present the construction of free commutative f-idempotent groupoids, where f is an irreducible element in $(E, \circ, e)$ and $|\, f\,| \geq 3$. Note that, if $f \in E \setminus E_c$, then f should be replaced with the "corresponding" groupoid power $\overline{f} \in E_c$ determined by the homomorphism $\psi : \mathbf{E} \to \mathbf{E}_c$ such that $\psi(e) = e$. Put $\psi(f) = \overline{f}$ for any $f \in E$. Thus, for $f = f_1 f_2$ we obtain that $\overline{f} = \overline{f_1} \odot \overline{f_2}$. The justification for the replacement of $f \in E \setminus E_c$ by $\overline{f} \in E_c$ is the commutativity of every groupoid in $\mathcal{V}^{fc}$. For instance, let $\boldsymbol{G} \in \mathcal{V}^{fc}$ and let $f = (e^2e)e^2 \in E \setminus E_c$. The groupoid power f is irreducible, since $|\, f\,| = 5$ is a prime number ([6]). Then, for every $a \in G$, $f^{\boldsymbol{G}}(a) = a$, i.e. $(a^2a)a^2 = a$. Since $\boldsymbol{G}$ is commutative, $a = (a^2a)a^2 = a^2(a^2a) = a^2(aa^2) = \overline{f}(a)$, where $\overline{f} = e^2(ee^2) \in E_c$. Since every $f \in E \setminus E_c$ can be replaced by $\overline{f} \in E_c$, we assume that $f \in E_c$.

Define the carrier of the free groupoid R_{fc} in $\mathcal{V}^{fc}$ by:

$$R_{fc} = \{t \in T_c : (\forall u \in T_c) \;\; f(u) \notin P(t)\}.$$

Directly from the definition of R_{fc} we obtain

Proposition 4.1 *a)* $X \subseteq R_{fc} \subseteq T_c$; $\;\; t \in R_{fc} \;\Rightarrow\; P(t) \subseteq R_{fc}$.
b) $t, u \in R_{fc} \;\Rightarrow\; [\, t \odot u \in R_{fc} \;\Leftrightarrow\; (\forall v \in R_{fc})\, t \odot u \neq f(v)\,]$.
c) $t, u \in R_{fc} \;\Rightarrow\; [\, t \odot u \notin R_{fc} \;\Leftrightarrow\; (\exists v \in R_{fc})\, t \odot u = f(v)\,]$.

Proof. a) and c) are clear.
b) If $t \odot u \in R_{fc}$, then by the definition of R_{fc}, $f(v) \notin P(t \odot u)$ for every $v \in T_c$, and thus $f(v) \notin \{t \odot u\}$, i.e. $f(v) \neq t \odot u$. Since $T_c \supseteq R_{fc}$, it follows that $v \in R_{fc}$. The converse is obvious.

Define an operation $*$ on R_{fc} by:

$$t, u \in R_{fc} \Rightarrow t * u = \begin{cases} t \odot u, & \text{if } t \odot u \in R_{fc} \\ v, & \text{if } t \odot u = f(v) \text{ for some } v \in R_{fc}. \end{cases}$$

Clearly, $t * u$ is a uniquely determined element of R_{fc} when $t \odot u \in R_{fc}$. If $t \odot u \notin R_{fc}$, then by Prop.4.1 c), there is $v \in R_{fc}$, such that $t \odot u = f(v)$. If there is $v_1 \in R_{fc}$ such that $t \odot u = f(v_1)$, then $f(v) = f(v_1)$. By Prop.1.1 b), it follows that $v_1 = v$. Thus, $v = t * u$ is a uniquely determined element of R_{fc}. Hence, $\boldsymbol{R}_{fc} = (R_{fc}, *)$ is a groupoid.

$\boldsymbol{R}_{fc}$ is a commutative f-idempotent groupoid. Namely, denote by f_* the interpretation of f in $\boldsymbol{R}_{fc}$, defined by:

$$e_*(t) = t \text{ and } (f_1 \odot f_2)_*(t) = (f_1)_*(t) * (f_2)_*(t), \text{ if } f = f_1 \odot f_2 \in E_c.$$

We will show that $t * u = u * t$ and that $f_*(w) = w$, for every $t, u, w \in R_{fc}$. If $t \odot u \in R_{fc}$, then $t * u = t \odot u = u \odot t = u * t$. If $t \odot u \notin R_{fc}$, then $t * u = v$

for some $v \in R_{fc}$, such that $t \odot u = f(v)$. However, $t \odot u = u \odot t$ in T_c and so $u \odot t = f(v)$. Thus, $u * t = v = t * u$. Let $f = f_1 f_2$ and $w \in R_{fc}$. Since f_1, f_2 are proper subterms of f it follows that $(f_i)_*(w) = f_i(w)$, for $i = 1, 2$. Therefore, $f_*(w) = (f_1 \odot f_2)_*(w) = (f_1)_*(w) * (f_2)_*(w) = f_1(w) * f_2(w) = w$. Hence, $\boldsymbol{R}_{fc} \in \mathcal{V}^{fc}$.

By induction on the length of terms one can show that X is the smallest generating set for $\boldsymbol{R}_{fc}$.

The universal mapping property for $\boldsymbol{R}_{fc}$ in $\mathcal{V}^{fc}$ over X holds. Namely, let $\boldsymbol{G} = (G, \cdot) \in \mathcal{V}^{fc}$ and $\lambda : X \to G$ is a mapping. Denote by φ the homomorphism from $\mathbf{T}_c$ into $\boldsymbol{G}$ that is an extension of λ and put $\psi = \varphi|_{R_{fc}}$. It suffices to show that $\varphi(t * u) = \varphi(t)\varphi(u)$, for any $t, u \in R_{fc}$. If $t \odot u \in R_{fc}$, then $t * u = t \odot u$, so $\varphi(t * u) = \varphi(t \odot u) = \varphi(t)\varphi(u)$. If $t \odot u \notin R_{fc}$, then $t * u = v$, where $v \in R_{fc}$ and $t \odot u = f(v)$. Thus $\varphi(t * u) = \varphi(v) = [\boldsymbol{G} \in \mathcal{V}^{fc}] = f(\varphi(v)) = [\,\varphi \text{ is a homomorphism}\,] = \varphi(f(v)) = \varphi(t \odot u) = \varphi(t)\varphi(u)$. Hence, $\psi = \varphi|_{R_{fc}}$ is a homomorphism from $\boldsymbol{R}_{fc}$ into $\boldsymbol{G}$ such that ψ is an extension of λ. Therefore, $\boldsymbol{R}_{fc} = (R_{fc}, *)$ is $\mathcal{V}^{fc}$-free groupoid over X.

Note that $\boldsymbol{R}_{fc} = (R_{fc}, *)$ is a cancellative groupoid. Namely, let $t, u, v \in R_{fc}$ and let $t * u = t * v$. If $t \odot u, t \odot v \in R_{fc}$, then $t \odot u = t \odot v$ in $\mathbf{T}_c$. Since the groupoid $\mathbf{T}_c$ is injective, i.e. $x_1 \odot x_2 = y_1 \odot y_2 \Rightarrow \{x_1, x_2\} = \{y_1, y_2\}$, one obtains that $\{t, u\} = \{t, v\}$, i.e. $u = v$. If $t \odot u$, $t \odot v \notin R_{fc}$, then $t \odot u = f(w_1)$, $t \odot v = f(w_2)$, for some $w_1, w_2 \in R_{fc}$. However, $w_1 = t * u = t * v = w_2$, and thus $f(w_1) = f(w_2)$. Therefore, $t \odot u = t \odot v$, i.e. $u = v$. If $t \odot u \in R_{fc}$, $t \odot v \notin R_{fc}$, then $t * u = t * v$ is not possible in $\boldsymbol{R}_{fc}$. Namely, $t * u = t \odot u$ and $t * v = w$, for some $w \in R_{fc}$ such that $t \odot v = f(w)$. Thus, $t \odot u = t * u = t * v = w$. Using this, one obtains that $t \odot v = f(t \odot u)$, i.e. $t \odot v = (f_1 \odot f_2)(t \odot u) = f_1(t \odot u) \odot f_2(t \odot u)$ in $\mathbf{T}_c$ which implies that $\{t, v\} = \{f_1(t \odot u), f_2(t \odot u)\}$. Hence, $t = f_1(t \odot u)$ or $t = f_2(t \odot u)$. None of this is possible, since $|\,t\,| \neq |\,f_i\,| \cdot (|\,t\,| + |\,u\,|)$. Symmetrically for the case $t \odot u \notin R_{fc}$, $t \odot v \in R_{fc}$. Thus, $\boldsymbol{R}_{fc}$ is left cancellative, and by commutativity it is right cancellative.

Theorem 4.2 *The groupoid $\boldsymbol{R}_{fc} = (R_{fc}, *)$ is free in $\mathcal{V}^{fc}$ over X and it is cancellative.*

Now, consider the monoid of powers for the variety $\mathcal{V}^{fc}$. Put

$$E_{fc} = \{g \in E_c : (\forall h \in E_c)\ f(h) \notin P(g)\}.$$

Define an operation "$\circ$" on E_{fc} according to (1), as a restriction of the operation "$\circ$" in $(E_c, \circ, e)$. By a direct verification one can show that analogous propositions like Prop.3.1–Prop.3.5 hold, putting $(E_{fc}, \circ, e)$ instead of $(E_f, \circ, e)$ and $(E_c, \circ, e)$ instead of $(E, \circ, e)$. Thus,

Theorem 4.3 *For every irreducible element $f \in E_c$, $|\,f\,| \geq 3$, the monoid $(E_{fc}, \circ, e)$ is free over a countable set of irreducible elements in $(E_c, \circ, e)$.*

The answer of (A) and of (B) is positive for $(E_f, \circ, e)$ and $(E_{fc}, \circ, e)$.

References

[1] V. Celakoska-Jordanova, *Free groupoids in the class of power left and right idempotent groupoids*, International Journal of Algebra, Vol.2, (2008), **no. 10**, 451 - 461

[2] V. Celakoska-Jordanova, *Free power-commutative groupoids*, Math. Slovaca (accepted).

[3] Ǵ. Čupona, N. Celakoski, *On groupoids with the identity* $xy^2 = xy$, Contributions Sec. Math. Tech. Sci., MANU, **XVIII, 1–2** (1997), 5 – 16.

[4] Ǵ. Čupona, N. Celakoski, *On groupoids with the identity* $x^2y^2 = xy$, Contributions Sec. Math. Tech. Sci., MANU, **XVIII, 1–2** (1997), 5 – 15.

[5] Ǵ. Čupona, S. Ilić, *Free groupoids with* $x^n = x$*, II*, Novi Sad J.Math., Vol.**29**, No.1. (1999), 147 – 154.

[6] Ǵ. Čupona, N. Celakoski, S. Ilić, *Groupoid powers*, Bulletin Mathématique, **25 (LI)** Skopje, R. Macedonia, (2001), 5 – 12.

[7] Ǵ. Čupona, N. Celakoski, S. Ilić, *On monoassociative groupoids*, Bulletin Mathématique, **26 (LII)** (2002), 5-16.

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