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On Quaternary Associative Operations*

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Abstract. Some properties of quaternary associative operations that have neutral sequences are investigated. A characterization of such operations by means of binary primitive operations is established.

1. Preliminary notes

In the paper [1] ternary associative operations that have neutral sequences are considered. Following this note, in the present work we investigate some properties of quaternary associative operations that have neutral sequences, and we obtain a characterization of such operations by means of their primitive binary operations. The main results of the paper are Theorem 2.1 and Theorem 3.1.

For the sake of completeness of the exposition some definitions and notations that will be used further on are stated bellow.

Let M be a nonempty set and n be a positive integer. An *n-ary operation A* on M is a mapping from M^n into M, where M^n is the n-fold direct power of M. The pair (M, A) is called an *n-groupoid*. The elements of M^n, i.e. the n-tuples $(x_1, x_2, \dots, x_n)$, where $x_i \in M$, $i \in \{1, 2, \dots, n\}$, will be denoted by $x_1x_2 \dots x_n$ or x_1^n and will be referred as *sequences*. In the case when $x_1 = x_2 = \dots x_n = x$, the sequence x_1^n is denoted by $\bar{x}^n$. If A maps the element $x_1^n \in M^n$ into $y \in M$, then we write $A(x_1^n) = y$ or $[x_1^n] = y$.

In the special cases when $n = 1, 2, 3$ or 4, the operation is said to be *unary*, *binary*, *ternary* or *quaternary*, respectively.

Let A be an n-ary operation on M. An m-ary operation B on M is said to be *induced* by A if the following conditions are satisfied:

(i) if $m = 1$, then $B(x) = x$, for any $x \in M$;
(ii) if $m > 1$, then there are m_i-ary operations $B_1, B_2, \dots, B_n$ on M, induced by A, such that

$$B(x_1^m) = A(B_1(x_1^{m_1}), B_2(x_{m_1+1}^{m_1+m_2}), \dots, B_n(\dots, x_m)),$$

where $m = m_1 + m_2 + \cdots + m_n$.

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If B is induced by A, than A is called a *primitive operation* for B.

By induction on m one can show that $m = k(n-1)+1$, for some nonnegative integer k.

A sequence e_1^{n-1} is said to be *i-neutral* for A ($i \in \{1, 2, \dots, n\}$) if for any $x \in M$, $A(e_1^{i-1} x e_i^{n-1}) = x$; and, e_1^{n-1} is called *neutral sequence* for A if it is i-neutral for all $i = 1, 2, \dots, n$. (As usual, x_k^k denotes the element x_k and x_k^{k-1} the empty symbol.)

An element $e \in M$ is said to be *i-neutral* for A ($i \in \{1, 2, \dots, n\}$) if the sequence $\bar{e}^{n-1}$ is i-neutral for A; and, e is *neutral* for A if it is i-neutral for all $i = 1, 2, \dots, n$.

An element $c \in M$ is called *central element* for the n-groupoid (M, A) if for all $x_1, \dots, x_{n-1} \in M$ and all i, j such that $1 \le i < j \le n$, the following equalities hold:

$$A(x_1^{i-1} c x_i^{n-1}) = A(x_1^{j-1} c x_j^{n-1}).$$

The set of all central elements in (M, A) is called the *center* ([2]) of (M, A) and is denoted by $Z(M)$.

An n-ary operation A on a set M is said to be *associative* if for each $i = 2, 3, \dots, n$ and for any $x_1, x_2, \dots, x_{2n-1} \in M$, the following equality holds:

$$A(A(x_1^n) x_{n+1}^{2n-1}) = A(x_1^{i-1} A(x_i^{i+n-1}) x_{i+n}^{2n-1}).$$

Specially, if A is a quaternary operation, then A is associative if

$$A(A(x_1^4) x_5^7) = A(x_1 A(x_2^5) x_6^7) = A(x_1^2 A(x_3^6) x_7) = A(x_1^3 A(x_4^7)).$$

It is easily shown that if A is a quaternary associative operation, then there is only one $(3k+1)$-ary operation B that is induced by A, for any nonnegative integer k.

In what follows we will assume that A is a quaternary operation on a set M and we will often write $[xyzt]$ instead of $A(xyzt)$.

2. Quaternary associative operations with neutral sequences

A sequence $(e_1 e_2 e_3)$ that is 2-neutral and 3-neutral for a quaternary associative operation $[\,]$ is in fact i-neutral for all $i = 1, 2, 3, 4$ and each member e_i of $(e_1 e_2 e_3)$ is a central element for $[\,]$. This is the main claim of the following theorem.

Theorem 2.1. *Let $[\ ]$ be a quaternary associative operation on a set M and $(e_1 e_2 e_3)$ be 2-neutral and 3-neutral sequence for $[\,]$. Then:*

(a) *the sequences $(e_{\nu_1} e_{\nu_2} e_{\nu_3})$ are neutral, for any permutation $\nu_1 \nu_2 \nu_3$ of the numbers $1, 2, 3$;*
(b) *every member of the sequence $(e_1 e_2 e_3)$ is a central element for $(M, [\,])$;*
(c) *a sequence $(f_1 f_2 f_3)$ is 1-neutral if and only if $(f_1 f_2 f_3)$ is 4-neutral.*

Proof. (a) There are 24 cases that should be considered. We will show the following cases: 1-neutrality of the sequences $(e_1 e_2 e_3)$ and $(e_1 e_3 e_2)$, 4-neutrality of the sequences $(e_1 e_2 e_3)$ and $(e_2 e_3 e_1)$, and 2-neutrality of $(e_1 e_3 e_2)$.

Using the fact that $(e_1e_2e_3)$ is 2-neutral and 3-neutral sequence for $[\,]$ and that $[\,]$ is an associative operation on M one can show that the sequences $(e_1e_2e_3)$ and $(e_1e_3e_2)$ are 1-neutral. Namely,

$$[xe_1e_2e_3] = [e_1e_2[xe_1e_2e_3]e_3] = [e_1e_2x[e_1e_2e_3e_3]] = [e_1e_2xe_3] = x$$
$$[xe_1e_3e_2] = [e_1e_2[xe_1e_3e_2]e_3] = [e_1e_2x[e_1e_3e_2e_3]] = [e_1e_2xe_3] = x.$$

4-neutrality of the sequence $(e_1e_2e_3)$ can be shown by using its 2-neutrality and 1-neutrality. Indeed,

$$[e_1e_2e_3x] = [e_1[e_1e_2e_3x]e_2e_3] = [[e_1e_1e_2e_3]xe_2e_3] = [e_1xe_2e_3] = x.$$

Using the 2-neutrality and 4-neutrality of $(e_1e_2e_3)$ one can show the 4-neutrality of the sequence $(e_2e_3e_1)$:

$$[e_2e_3e_1x] = [e_1[e_2e_3e_1x]e_2e_3] = [[e_1e_2e_3e_1]xe_2e_3] = [e_1xe_2e_3] = x.$$

2-neutrality of the sequence $(e_1e_3e_2)$ can be shown by the 4-neutrality of $(e_2e_3e_1)$, 2-neutrality of $(e_1e_2e_3)$ and 1-neutrality of $(e_1e_3e_2)$:

$$[e_1xe_3e_2] = [e_1x[e_2e_3e_1e_3]e_2] = [[e_1xe_2e_3]e_1e_3e_2] = [xe_1e_3e_2] = x.$$

The other cases can be easily shown similarly as the cases proved above.

(b) It suffices to show that $[e_ixyz] = [xe_iyz] = [xye_iz] = [xyze_i]$, for all $x, y, z \in M$ and $i = 1, 2, 3, 4$. We will state the proof only for $i = 1$, using 1-neutrality of the sequence $(e_2e_3e_1)$ and 2-neutrality of $(e_1e_2e_3)$. Indeed,

$$\begin{aligned}[e_1xyz] &= [e_1[xe_2e_3e_1]yz] = [[e_1xe_2e_3]e_1yz] = [xe_1yz] = \\ &= [xe_1[ye_2e_3e_1]z] = [x[e_1ye_2e_3]e_1z] = [xye_1z] = \\ &= [xye_1[ze_2e_3e_1]] = [xy[e_1ze_2e_3]e_1] = [xyze_1].\end{aligned}$$

The other cases can be shown similarly, using (a).

(c) Let $(f_1f_2f_3)$ be 1-neutral sequence. By (b), for any $x \in M$,

$$\begin{aligned}[f_1f_2f_3x] &= [f_1f_2f_3[e_1xe_2e_3]] = [[f_1f_2f_3e_1]xe_2e_3] = \\ &= [[e_1f_1f_2f_3]xe_2e_3] = [e_1xe_2e_3] = x.\end{aligned}$$

Therefore, the sequence $(f_1f_2f_3)$ is 4-neutral. Conversely, let $(f_1f_2f_3)$ be 4-neutral sequence. By (b) one obtains:

$$\begin{aligned}[xf_1f_2f_3] &= [[e_1e_2xe_3]f_1f_2f_3] = [e_1e_2x[e_3f_1f_2f_3]] = \\ &= [e_1e_2x[f_1f_2f_3e_3]] = [e_1e_2xe_3] = x.\end{aligned}$$

Therefore, the sequence $(f_1f_2f_3)$ is 1-neutral. □

By E_i we denote the subset of M that consists of all i-neutral elements for $[\,]$, i.e. $e \in E_i$ if and only if e is i-neutral for $[\,]$, $i \in \{1, 2, 3, 4\}$.

The following two propositions are corollaries of Theorem 2.1.

Corollary 2.1. *Let* $[\,]$ *be a quaternary associative operation on* M *and* $(e_1e_2e_3)$ *be* 2*-neutral and* 3*-neutral sequence for* $[\,]$. *If* $e \in E_2 \cap E_3$, *then* $E_2 \cap E_3 \subseteq E_1 = E_4$.

Proof. Let $e \in E_2 \cap E_3$. Then by Theorem 2.1 it follows that the sequence (eee) is i-neutral, for $i = 1, 2, 3, 4$. Hence, $e \in E_1$ and $e \in E_4$, i.e. $e \in E_1 \cap E_2 \cap E_3 \cap E_4$. Therefore, $E_2 \cap E_3 \subseteq E_1$. On the other hand, by Theorem 2.1 (c), it follows that $e \in E_1$ if and only if $e \in E_4$, i.e. $E_1 = E_4$. Hence, $E_2 \cap E_3 \subseteq E_1 = E_4$. □

Corollary 2.2. *Let $(M, \cdot)$ be a semigroup and let for any $x \in M$, $a \cdot b \cdot x \cdot c = x = a \cdot x \cdot b \cdot c$, for some $a, b, c \in M$. Then $a \cdot b \cdot c = a \cdot c \cdot b = b \cdot a \cdot c = b \cdot c \cdot a = c \cdot a \cdot b = c \cdot b \cdot a$ is a neutral element for the semigroup $(M, \cdot)$ and $a, b, c \in Z(M)$.*

Proof. Putting $[xyzt] = x \cdot y \cdot z \cdot t$, we obtain a quaternary associative operation, for which the sequence (abc) is 2-neutral and 3-neutral. By Theorem 2.1 it follows that the sequences (abc), (acb), (bac), (bca), (cab), (cba) are neutral for $[\ ]$. By 1-neutrality and 4-neutrality of the sequence (abc), it follows that for any $x \in M$, $x \cdot a \cdot b \cdot c = [xabc] = x$ and $a \cdot b \cdot c \cdot x = [abcx] = x$. Thus $a \cdot b \cdot c$ is a neutral element for $(M, \cdot)$. The same is true for all sequences obtained by permuting (abc). However, if a semigroup posses a neutral element, then it is uniquely determined, and thus $a \cdot b \cdot c = a \cdot c \cdot b = b \cdot a \cdot c = b \cdot c \cdot a = c \cdot a \cdot b = c \cdot b \cdot a$ is a neutral element for the semigroup $(M, \cdot)$. Using this, one can see that: $x \cdot a = (a \cdot b \cdot c) \cdot x \cdot a = a \cdot (b \cdot c \cdot x \cdot a) = a \cdot [bcxa] = a \cdot x$ for all $x \in M$, i.e. $a \in Z(M)$. The same is true for b and c. □

Note that an n-ary operation $(n \geq 3)$ on a set M may have more than one neutral element ([3]).

Remark 1. 2-neutrality and 3-neutrality in Theorem 2.1 can not be replaced by 1-neutrality and 4-neutrality of the sequence $(e_1 e_2 e_3)$, see Example 2.1 below. Also, the associativity of the quaternary operation can not be replaced by 2-neutrality or 3-neutrality, see Example 2.3.

Example 2.1. Let D_4 be the group of symmetries of the square. We are using ρ_i for *rotations*, μ_i for *mirror images* in perpendicular bisectors of sides, and δ_i for *diagonal flips*. The table for D_4 is given bellow.

	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_1	ρ_1	ρ_2	ρ_3	ρ_0	δ_2	δ_1	μ_1	μ_2
ρ_2	ρ_2	ρ_3	ρ_0	ρ_1	μ_2	μ_1	δ_2	δ_1
ρ_3	ρ_3	ρ_0	ρ_1	ρ_2	δ_1	δ_2	μ_2	μ_1
μ_1	μ_1	δ_1	μ_2	δ_2	ρ_0	ρ_2	ρ_1	ρ_3
μ_2	μ_2	δ_2	μ_1	δ_1	ρ_2	ρ_0	ρ_3	ρ_1
δ_1	δ_1	μ_2	δ_2	μ_1	ρ_3	ρ_1	ρ_0	ρ_2
δ_2	δ_2	μ_1	δ_1	μ_2	ρ_1	ρ_3	ρ_2	ρ_0

Define a quaternary operation on D_4 by $[xyzt] = x \cdot y \cdot z \cdot t$. By the above table it can be easily seen that the sequences $(\rho_1\rho_1\rho_2)$, $(\rho_2\rho_3\rho_3)$, $(\rho_1\mu_2\delta_1)$ are 1-neutral and 4-neutral, but they are not 2-neutral or 3-neutral. For example, $[\rho_1 x \rho_1 \rho_2] \neq x$ if $x = \rho_2$. The sequence $(\rho_0\rho_2\rho_2)$ is 1-neutral, 2-neutral and 4-neutral, but it is not 3-neutral. The sequence $(\mu_1\mu_1\rho_0)$ is 1-neutral, 3-neutral and 4-neutral, but it is not 2-neutral. ||

The following example provides sequences that are 2-neutral and 3-neutral.

Example 2.2. Let G be the group that is the direct product of the groups D_4 and $\mathbb{Z}_4$, i.e. $G = D_4 \times \mathbb{Z}_4$. Define a quaternary operation on G by $[xyzt] = x \cdot y \cdot z \cdot t$. One can obtain neutral sequences from the elements $a_0 = (\rho_0, 0)$, $a_1 = (\rho_0, 1)$, $a_2 = (\rho_0, 2)$, $a_3 = (\rho_0, 3)$, $b_0 = (\rho_2, 0)$, $b_1 = (\rho_2, 1)$, $b_2 = (\rho_2, 2)$, $b_3 = (\rho_2, 3)$. For example, $(a_0a_1a_3)$, $(a_1a_1a_2)$, $(a_1b_1b_2)$, $(a_1b_0b_3)$, $(a_2a_3a_3)$, $(a_3b_2b_3)$, are 2-neutral and 3-neutral sequences, and thus by Theorem 2.1, they are i-neutral for all $i = 1, 2, 3, 4$. ||

Example 2.3. Define a quaternary operation on $M = \{a, b\}$ by: $a = [aaaa] = [aaab] = [aaba] = [abaa] = [baaa] = [bbaa] = [bbba] = [bbbb]$, $b = [baba] = [baab] = [abab] = [abba] = [aabb] = [abbb] = [babb] = [bbab]$. The sequence (aba) is i-neutral, $i = 1, 2, 3, 4$, but the operation is not an associative one. Namely, $[a[aaab]ba] = [aaba] = a \neq b = [abba] = [[aaaa]bba]$. ||

We will state two lemmas that will be used further on for proving Proposition 2.1.

Lemma 2.1. *Let [ ] be a quaternary associative operation on M and $(e_1e_2e_3)$ be 2-neutral and 3-neutral sequence for [ ]. Then, for any positive integers r and s, the following three conditions are equivalent:*
(a) $[x_1^{3r}e_1] = [y_1^{3s}e_1]$; *(b)* $[x_1^{3r}e_2] = [y_1^{3s}e_2]$; *(c)* $[x_1^{3r}e_3] = [y_1^{3s}e_3]$.

(We denote by the same symbol [ ] the $(3k+1)$-ary operation induced by the quaternary operation [ ].)

Proof. $(a) \Rightarrow (b)$. Let $[x_1^{3r}e_1] = [y_1^{3s}e_1]$. Then, by the associativity of [ ] and 2-neutrality of the sequence $(e_1e_2e_3)$, we obtain: $[x_1^{3r}e_2] = [x_1^{3r}[e_1e_2e_2e_3]] = [[x_1^{3r}e_1]e_2e_2e_3] = [[y_1^{3s}e_1]e_2e_2e_3] = [y_1^{3s}[e_1e_2e_2e_3]] = [y_1^{3s}e_2]$.

The statements $(b) \Rightarrow (c)$ and $(c) \Rightarrow (a)$ can be easily shown in a similar way as $(a) \Rightarrow (b)$, using the associativity of [ ] and 2-neutrality of the sequences $(e_2e_3e_1)$ or $(e_3e_1e_2)$, respectively. □

By induction on k one can prove the following

Lemma 2.2. *Let [ ] be a quaternary associative operation on M and $(e_1e_2e_3)$ be 2-neutral and 3-neutral sequence for [ ]. Then, $y = [y\bar{e}_1^k\bar{e}_2^k\bar{e}_3^k]$, for any $y \in M$ and any positive integer k.* □

Proposition 2.1. *Let [ ] be a quaternary associative operation on M and let $(e_1e_2e_3)$ be 2-neutral and 3-neutral sequence for [ ]. Then, for any $k \geq 1$:*

(a) $[x\bar{e_1}^{3k}] = y$ *if and only if* $x = [y\bar{e_2}^{3k}\bar{e_3}^{3k}]$;
(b) $[x\bar{e_2}^{3k}] = y$ *if and only if* $x = [y\bar{e_1}^{3k}\bar{e_3}^{3k}]$;
(c) $[x\bar{e_3}^{3k}] = y$ *if and only if* $x = [y\bar{e_1}^{3k}\bar{e_2}^{3k}]$.

Proof. The statements (a), (b), (c) are shown by using Lemma 2.2, Theorem 2.1 (b) and Lemma 1.1. We will show only the statement (a), since (b) and (c) can be shown similarly. Indeed,

$$\begin{aligned} [x\bar{e_1}^{3k}] = y \quad &\Leftrightarrow \quad [x\bar{e_1}^{3k}] = [y\bar{e}_1^k\bar{e}_2^k\bar{e}_3^k] \quad \Leftrightarrow \\ \Leftrightarrow \quad [x\bar{e_1}^{3k-1}e_1] = [y\bar{e}_1^{k-1}\bar{e}_2^k\bar{e}_3^k e_1] \quad &\Leftrightarrow \quad [x\bar{e_1}^{3k-1}e_2] = [y\bar{e}_1^{k-1}\bar{e}_2^k\bar{e}_3^k e_2]. \end{aligned}$$

Repeating this procedure $k-1$ times we obtain: $[x\bar{e_1}^{2k}\bar{e}_2^k] = [y\bar{e}_2^k\bar{e}_3^k\bar{e}_2^k]$. By Theorem 2.1 (b), $[x\bar{e_1}^{2k}\bar{e}_2^k] = [y\bar{e}_2^{2k}\bar{e}_3^k]$. By this, using Lemma 2.2, Theorem 2.1 (b) and Lemma 1.1, we obtain:

$$\begin{aligned} [x\bar{e_1}^{2k}\bar{e}_2^k] = [[y\bar{e}_1^k\bar{e}_2^k\bar{e}_3^k]\bar{e}_2^k\bar{e}_3^k\bar{e}_2^k] \quad &\Leftrightarrow \quad [x\bar{e_1}^{2k}\bar{e}_2^k] = [y\bar{e}_1^k\bar{e}_2^{3k}\bar{e}_3^{2k}] \quad \Leftrightarrow \\ \Leftrightarrow \quad [x\bar{e_1}^{2k-1}\bar{e}_2^k e_1] = [y\bar{e}_1^{k-1}\bar{e}_2^{3k}\bar{e}_3^{2k} e_1] \quad &\Leftrightarrow \quad [x\bar{e_1}^{2k-1}\bar{e}_2^k e_3] = [y\bar{e}_1^{k-1}\bar{e}_2^{3k}\bar{e}_3^{2k} e_3]. \end{aligned}$$

Repeating this procedure $k-1$ times we obtain that $[x\bar{e_1}^{k}\bar{e}_2^k\bar{e}_3^k] = [y\bar{e}_2^{3k}\bar{e}_3^{3k}]$. By Lemma 2.2, $x = [y\bar{e}_2^{3k}\bar{e}_3^{3k}]$. □

3. Primitive operations for a given quaternary operation

For every n-ary operation A, the "identity" unary operation and A itself are primitive operations; they are the only ones unary and n-ary operations that are primitive for A. If $n = 3$, then the rest of the primitive operations are binary. The same is true for $n = 4$, i.e. the rest of the primitive operations are binary (they cannot be ternary because there is no $k \in \mathbb{N}$, such that $4 = k \cdot (3-1) + 1$). If B is a binary operation, then the quaternary operations induced by B are: $[xyzt]_1 = B(B(B(xy)z)t)$, $[xyzt]_2 = B(B(xB(yz))t)$, $[xyzt]_3 = B(B(xy)B(zt))$, $[xyzt]_4 = B(xB(B(yz)t))$, $[xyzt]_5 = B(xB(yB(zt)))$.

The following question arises naturally: which primitive operations for a quaternary associative operation that has 2-neutral and 3-neutral sequence are associative? Before we give the answer to this question, we will introduce some notations.

Let $[\ ]$ be a quaternary operation on a set M and let $a \in M$ be a fixed element. We define two binary and two ternary operations on M by:

$$\begin{aligned} A_{aa}(xy) = [xyaa] \quad &\text{and} \quad {}_{aa}A(xy) = [aaxy], \\ A_a(xyz) = [xyza] \quad &\text{and} \quad {}_aA(axyz) = [axyz] \end{aligned}$$

for any $x, y, z \in M$.

In a similar way one can define binary operations as follows:

$$\begin{aligned} A_a^a(xy) = [xaay], \qquad &{}_a^aA(xy) = [axya], \\ {}_aA^a(xy) = [axay], \qquad &{}^aA_a(xy) = [xaya]. \end{aligned}$$

The following two propositions provide necessary and sufficient conditions for a binary operation to be primitive for a quaternary associative operation.

Proposition 3.1. *Let $[\ ]$ be a quaternary associative operation on a set M such that $(e_1e_2e_3)$ is a 1-neutral sequence for $[\ ]$ and B be a binary operation on M.*

Then B is primitive for [ ], $[xyzt] = B(xB(yB(zt)))$, if and only if there is a 1-neutral element $e \in M$ such that $B(xy) = A_{ee}(xy)$.

Proof. Let $[xyzt] = B(xB(yB(zt)))$. Then, for any $x \in M$, $x = [xe_1e_2e_3] = B(xB(e_1B(e_2e_3))) = B(xe)$, where $e = B(e_1B(e_2e_3))$. The element e is 1-neutral for [ ]. Namely, for any $x \in M$, $[xeee] = B(xB(eB(ee))) = B(xe) = x$. Then, $B(xy) = B(xB(ye)) = B(xB(yB(ee))) = [xyee]$. Conversely, by the associativity of [ ] and 1-neutrality of the element e, for any $x, y, z, t \in M$, we obtain:

$$B(xB(yB(zt))) = [x[y[ztee]ee]ee] = [x[[yzte]eee]ee] = \\ = [x[yzte]ee] = [[xyzt]eee] = [xyzt].$$ □

The following proposition can be proved in a similar way as Proposition 3.1.

Proposition 3.2. *Let [ ] be a quaternary associative operation on a set M such that $(e_1e_2e_3)$ is a 4-neutral sequence for [ ] and B be a binary operation on M. Then B is primitive for [ ], $[xyzt] = B(B(B(xy)z)t)$, if and only if there is a 4-neutral element $e \in M$ such that $B(xy) = {}_{ee}A(xy)$.* □

The following example shows that the associativity of the quaternary operation in Proposition 3.1 and 3.2 can not be omit.

Example 3.1. By example 2.1, the sequence $(\rho_1\mu_2\delta_1)$ is 1-neutral for the quaternary associative operation $[xyzt] = x \cdot y \cdot z \cdot t$ and ρ_0 is 1-neutral element for [ ]. The binary operation B, $B(xy) = x \cdot y$, is primitive for [ ], i.e. $B(x(B(yB(zt))) = [xyzt]$. Then $\rho_0 = B(\rho_1B(\mu_2\delta_1))$ and $B(xy) = [xy\rho_0\rho_0]$. However, this is not true if [ ] is not associative. For example, let $M = \mathbb{Z}_4$. Define a quaternary operation on M by $[xyzt] = x + 2y$. This operation is not associative. The binary operation $B(xy) = x + 2y$ is primitive for [ ], $[xyzt] = B(xB(yB(zt)))$. The sequence (213) is 1-neutral for [ ] and the 1-neutral element 2 is not equal to $B(2B(13))$, since $B(2B(13)) = B(23) = 0$. ‖

Remark 2. By the properties of associative binary operations it follows that a quaternary operation on M is associative if at least one of its primitive binary operations is associative. The converse does not hold, i.e. from the associativity of the quaternary operation it does not follow the associativity of all of its primitive binary operations. For example, let $M = \mathbb{Z}_4$. Define a binary operation B and a quaternary operation [ ] on M by: $B(xy) = 2x + 2y$ and $[xyzt] = 0$, for all $x, y, z, t \in M$. The operation B is not associative, [ ] is associative and B is primitive for [ ], $[xyzt] = B(B(xy)B(zt))$.

Theorem 3.1. *Let [ ] be a quaternary associative operation on a set M and $(e_1e_2e_3)$ be a 2-neutral and 3-neutral sequence for [ ].*

(a) *B is a primitive binary operation for A if and only if there is an element $e \in E$ (where $E = E_1 = E_4$ in Corollary 2.1) such that $B = A_{ee}$ or $B = {}_{ee}A$.*

(b) *If $e \in E$, then:*

$${}_{ee}A(xy) = A_{ee}(xy) \;\Leftrightarrow\; e \in E_2 \cap E_3 \;\Leftrightarrow\; \textit{the operation } A_{ee} \textit{ is associative.}$$

(c) *If* $e, f \in E$, *then:*

$$A_f = A_e \quad \Leftrightarrow \quad e = f$$
$${}_fA = A_e \quad \Leftrightarrow \quad e = f \in E_3.$$

Proof. (a) By Theorem 2.1, it follows that the sequence $(e_1e_2e_3)$ is 1-neutral and 4-neutral. Therefore, by Proposition 3.1, there is an element $e \in E$ such that $B = A_{ee}$ or, by Proposition 3.2, $B = {}_{ee}A$. The converse is also shown in Proposition 3.1 and 3.2.

(b) Let $A_{ee}(xy) = {}_{ee}A(xy)$. Then, for any $x \in M$, $x = [xeee] = A_{ee}(xe) = {}_{ee}A(xe) = [eexe]$, i.e. $e \in E_3$. Also, $x = [eeex] = {}_{ee}A(ex) = A_{ee}(ex) = [exee]$, i.e. $e \in E_2$. Therefore, $e \in E_2 \cap E_3$. Conversely, let $e \in E_2 \cap E_3$. Then,

$$\begin{aligned} {}_{ee}A(xy) &= [eexy] = [ee[exee]y] = [[eeex]eey] = \\ &= [xeey] = [xee[yeee]] = [x[eeye]ee] = [xyee] = A_{ee}(xy). \end{aligned}$$

Hence, $A_{ee}(xy) = {}_{ee}A(xy)$ if and only if $e \in E_2 \cap E_3$.

Let $e \in E_2 \cap E_3$. Then, for any $x, y, z \in M$,

$$\begin{aligned} A_{ee}(xA_{ee}(yz)) &= [x[yzee]ee] = [xy[zeee]e] = [xyze] = \\ &= [xy[eeze]e] = [[xyee]zee] = A_{ee}(A_{ee}(xy)z), \end{aligned}$$

i.e. the operation A_{ee} is associative. Conversely, let A_{ee} be an associative operation, i.e. $A_{ee}(A_{ee}(xy)z) = A_{ee}(xA_{ee}(yz))$, i.e. $[[xyee]zee] = [x[yzee]ee]$, for any $x, y, z \in M$. Then, for any $y \in M$:

$$[eyee] = [e[eeey]ee] = (\text{by the associativity of } A_{ee}) = [[eeey]eee] = [yeee] = y,$$

i.e. $e \in E_2$. Also, for any $z \in M$,

$$[eeze] = [ee[zeee]e] = [e[ezee]ee] = [ezee] = z,$$

i.e. $e \in E_3$. Hence, the operation A_{ee} is associative if and only if $e \in E_2 \cap E_3$.

(c) Let $A_f = A_e$, i.e. $[xyzf] = [xyze]$. Then,

$$e = [e_1e_2e_3e] = [e_1e_2e_3f] = f.$$

The converse is obvious.

Let ${}_fA = A_e$, i.e. $[fxyz] = [xyze]$. Then,

$$f = [fe_1e_2e_3] = {}_fA(e_1e_2e_3) = A_e(e_1e_2e_3) = [e_1e_2e_3e] = e.$$

Also, for any $x \in M$,

$$x = [eeex] = {}_eA(eex) = A_eeex = [eexe],$$

i.e. $e \in E_3$. For the converse, let $e = f \in E_3$. Then,

$$\begin{aligned} {}_fA(xyz) &= [fxyz] = [f[ffxf]yz] = [[fffx]fyz] = [xfyz] = \\ &= [xf[ffyf]z] = [x[fffy]fz] = [xyfz] = [xyf[ffzf]] = \\ &= [xy[fffz]f] = [xyzf] = [xyze] = A_e(xyz). \end{aligned}$$

□

Remark 3. If we omit the supposition of associativity or existence of 2-neutral and 3-neutral sequence for a quaternary operation [ ], then primitive binary operations can also exist in the case when $E_1 = E_4 = \varnothing$. It can be seen in the following examples.

Example 3.2. Let M be at least two element set and $a \in M$. Define a quaternary operation [ ] on M by $[xyzt] = a$, for any $x, y, z, t \in M$. It is obvious that [ ] is an associative operation for which the binary operation $B(xy) = a$ is primitive. However, $E_1 = E_2 = E_3 = E_4 = \varnothing$.

Example 3.3. Let M be the set of rational numbers. Define a quaternary operation [ ] on M by $[xyzt] = x + 2y + 2z + 2t - 3$, for all $x, y, z, t \in M$. It can be easily checked that $E_1 = \{\frac{1}{2}\}$ and that $E_2 = E_3 = E_4 = \varnothing$. The binary operation B on M defined by $B(xy) = x + 2y - 1$ is primitive for [ ], $B(B(B(xy)z)t) = x + 2y + 2z + 2t - 3 = [xyzt]$. Note that the operations [ ] and B are not associative, and $B(xy) = [xyee]$ or $B(xy) = [xeye]$, if $e = \frac{1}{2}$.

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