

FREE GRUPOIDS WITH $x^2x^2 = x^3x^3$

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Abstract. A description of free objects in the variety \mathcal{V} of groupoids defined by the identity $x^2x^2 = x^3x^3$ is obtained. The following method is used: one of the sides of the identity is considered as "suitable" and the other as "unsuitable" one. First, the left-hand side x^2x^2 is chosen as "suitable" and the set of elements of F (F being an absolutely free groupoid with a basis B) containing no parts that have the form x^3x^3 is taken as a "candidate" for the carrier of the desired free object in \mathcal{V} . Continuing this procedure, a \mathcal{V} -free object is obtained. Another construction of \mathcal{V} -free object is obtained by choosing the right-hand side x^3x^3 as "suitable" one.

0. INTRODUCTION

First, we introduce some notations.

Throughout the paper, $F = (F, \cdot)$ will denote a given absolutely free groupoid²⁾ (i.e. groupoid free in the class of all groupoids) with the basis B . The following two properties characterize F ([1]; L.1.5):

- a) F is injective (i.e. $ab = cd \Rightarrow a = c, b = d$);
- b) The set B of primes³⁾ is nonempty and generates F .

For every $w \in F$, a set $P(w)$ (called the *set of parts* of w) and the *length* $|w|$ of w are defined by:

$$P(b) = \{b\}, \quad P(uv) = \{uv\} \cup P(u) \cup P(v), \quad |b| = 1, \quad |uv| = |u| + |v|,$$

for every $b \in B$ and $u, v \in F$.

The subject of this paper is a construction of free groupoids in the variety \mathcal{V} of groupoids defined by the identity

$$x^2x^2 = x^3x^3. \quad 4) \tag{0.1}$$

In order to construct \mathcal{V} -free objects (i.e. free objects in the variety \mathcal{V}) we will recall the corresponding procedure given in [2] for the variety \mathcal{V}_1 of groupoids defined by the identity

$$xx^2 = x^2x^2 \tag{0.2}$$

¹⁾Research supported in part by MANU within the project "Free algebraic structures". I am grateful to Professor Ćorgi Ćupona, the project manager.

²⁾Notions as: groupoid, free groupoid, homomorphism ... have the usual meanings.

³⁾In a groupoid $G = (G, \cdot)$, $a \in G$ is *prime* iff $a \neq xy$, for all $x, y \in G$.

⁴⁾Here, x^n is defined by: $x^1 = x$, $x^{k+1} = x^kx$.

Key words and phrases. groupoid, variety, free groupoid.

Namely, choose first x^2x^2 as the "suitable" (i.e. xx^2 as the "unsuitable") side of (0.2). As a "candidate" for the carrier of a \mathcal{V}_1 -free groupoid, define the set

$$R = \{t \in F : (\forall \alpha \in F) \alpha\alpha^2 \notin P(t)\},$$

and then define an operation $*$ on R by

$$t, u \in R \Rightarrow \{t * u = tu \text{ if } tu \in R \ \& \ t * u = (t^2)^2 \text{ if } u = t^2\}.$$

The obtained groupoid $\mathbf{R} = (R, *)$ is a \mathcal{V}_1 -free groupoid with the basis B .

Next, we choose xx^2 as the "suitable" (i.e. x^2x^2 as the "unsuitable") side and define a "candidate" for the carrier of a \mathcal{V}_1 -free groupoid to be the set

$$F_1 = \{t \in F : (\forall \alpha \in F) (\alpha^2)^2 \notin P(t)\},$$

and then an operation $*_1$ on F_1 by

$$t, u \in F_1 \Rightarrow \{t *_1 u = tu \text{ if } tu \in F_1 \ \& \ t *_1 u = \alpha\alpha^2 \text{ if } t = u = \alpha^2\}.$$

Then $\mathbf{F}_1 = (F_1, *_1)$ is a groupoid that is not in \mathcal{V}_1 . As a consequence of the identity $xx^2 = x^2x^2$, we come to a new identity $\alpha^2(\alpha\alpha^2) = (\alpha\alpha^2)^2$. This suggests a definition of a new "candidate" $\mathbf{F}_2 = (F_2, *_2)$:

$$F_2 = \{t \in F_1 : (\forall \alpha \in F_1) (\alpha\alpha^2)^2 \notin P(t)\},$$

We obtain that $\mathbf{F}_2 \notin \mathcal{V}_1$ and come to a new identity in \mathcal{V}_1 :

$$(\alpha\alpha^2)(\alpha^2(\alpha\alpha^2)) = ((\alpha^2(\alpha\alpha^2))^2).$$

Continuing this procedure, we see regularity in the consequences of (0.2) that suggests introducing a special kind of groupoid powers $x \mapsto x^{<n>}$ defined by:

$$x^{<0>} = x, \quad x^{<1>} = x^2, \quad x^{<k+2>} = x^{<k>}x^{<k+1>}. \quad (0.3)$$

Using this, we have: $(\alpha^2)^2 = (\alpha^{<1>})^2$, $(\alpha\alpha^2)^2 = (\alpha^{<2>})^2$ etc. and a sequence of groupoids $\mathbf{F}_n = (F_n, *_n)$, $n \geq 0$, defined by: $\mathbf{F}_0 = \mathbf{F} = (F, \cdot)$,

$$F_1 = \{t \in F_0 : (\forall \alpha \in F_0) (\alpha^{<1>})^2 \notin P(t)\},$$

$$F_n = \{t \in F_{n-1} : (\forall \alpha \in F_n) (\alpha^{<n>})^2 \notin P(t)\},$$

$t, u \in F_n \Rightarrow \{t *_n u = t *_n u \text{ if } t *_n u \in F_n \ \& \ t *_n u = \alpha^{<n+1>} \text{ if } t = u = \alpha^{<n>}\}$

The groupoids \mathbf{F}_n are not in \mathcal{V}_1 . However, the fact that $F \supseteq F_1 \cdots \supseteq F_n \supseteq \dots$ and that \mathbf{F}_n is "better" than \mathbf{F}_{n-1} enables us to define a carrier R' of a free object in \mathcal{V}_1 by:

$$R' = \{t \in F : (\forall \alpha \in F, k \geq 1) (\alpha^{<k>})^2 \notin P(t)\} \quad (= \bigcap \{F_n : n \geq 1\})$$

and an operation $*'$ on R' by:

$$t, u \in R' \Rightarrow \{t *' u = tu \text{ if } tu \in R' \ \& \ t *' u = \alpha^{<k+1>} \text{ if } t = u = \alpha^{<k>}, k \geq 1\}.$$

Then $\mathbf{R}' = (R', *')$ is a \mathcal{V}_1 -free groupoid with the basis B and it is isomorphic to \mathbf{R} .

We use below the same method for constructing free objects in the variety \mathcal{V} of groupoids with $x^2x^2 = x^3x^3$.

1. CONSTRUCTION OF \mathcal{V} -FREE OBJECTS BY CHOOSING x^2x^2 AS THE "SUITABLE SIDE"

Choosing the left-hand side of (0.1) as "suitable", we define the first "candidate" for the carrier of a \mathcal{V} -free groupoid by:

$$F_1 = \{t \in F : (\forall \alpha \in F) (\alpha^3)^2 \notin P(t)\} \quad (1.1)$$

By (1.1) we obtain:

- 1) $t, u \in F_1 \Rightarrow \{tu \notin F_1 \Leftrightarrow t = u \text{ is a cube}^5\}$
- 2) $t, u \in F_1 \Rightarrow \{tu \in F_1 \Leftrightarrow [t \neq u \text{ or } (t = u \text{ is not a cube})]\}$
- 3) $t^2 \in F_1 \Leftrightarrow \{t \in F_1 \ \& \ t \text{ is not a cube}\}$
- 4) $t^3 \in F_1 \Leftrightarrow t^2 \in F_1$

Define an operation $*_1$ on F_1 by:

$$t, u \in F_1 \Rightarrow t *_1 u = \begin{cases} tu, & \text{if } tu \in F_1 \\ (\alpha^2)^2, & \text{if } t = u = \alpha^3. \end{cases}$$

By a direct verification we obtain that $F_1 = (F_1, *_1)$ is a groupoid. However, the equality (0.1), which has the form here

$$(t *_1 t) *_1 (t *_1 t) = ((t *_1 t) *_1 t) *_1 ((t *_1 t) *_1 t) \quad (1.2)$$

is not satisfied in F_1 . Namely, for $t = \alpha^3$, the left-hand side of (1.2) is $((\alpha^2)^2)^2$ and the right-hand side is $((\alpha^2)^2 \alpha^3)^2$. Thus, $F_1 \notin \mathcal{V}$. Therefore, as a consequence of (1.2), we obtain that: $((\alpha^2)^2)^2 = ((\alpha^2)^2 \alpha^3)^2$ is an identity in \mathcal{V} .

This suggests a definition of a new "candidate" $F_2 = (F_2, *_2)$:

$$F_2 = \{t \in F_1 : (\forall \alpha \in F_1) ((\alpha^2)^2 \alpha^3)^2 \notin P(t)\},$$

$$t, u \in F_2 \Rightarrow t *_2 u = \begin{cases} t *_1 u, & \text{if } t *_1 u \in F_2 \\ ((\alpha^2)^2)^2, & \text{if } t = u = (\alpha^2)^2 \alpha^3. \end{cases}$$

Checking (1.2) (when $*_1$ is substituted by $*_2$), we obtain that $F_2 \in \mathcal{V}$ and one more identity in \mathcal{V} : $((\alpha^2)^2)^2 = (((\alpha^2)^2)^2 ((\alpha^2)^2 \alpha^3))^2$. Continuing this procedure, we can see "regularity" in the consequences of the identity (1.2). This suggests introducing the following notations:

$$\begin{aligned} x^{(0)} &= x, & x^{(k+1)} &= (x^{(k)})^2; \\ x^{[0]} &= x, & x^{[k+1]} &= x^{(k+1)} x^{[k]}. \end{aligned} \quad (1.3)$$

It is easily seen, by induction on n , that:

⁵ $a \in G$ is a *cube* in a grupoid $\mathbf{G} = (G, \cdot)$ iff $(\exists \alpha \in G) a = \alpha^3$; if \mathbf{G} is injective then α is unique.

Proposition 1.1. *If $G = (G, \cdot)$ is any groupoid, then for each $x \in G$ and $m, n \geq 0$:*

$$x^{(m+n)} = (x^{(m)})^{(n)}.$$

By induction on p , one can show the following propositions:

Proposition 1.2. *If $x, y \in F$ and $p, q \geq 0$, then:*

- a) $|x^{(p)}| = 2^p|x|$; b) $|x^{[p]}| = (2^{p+1} - 1)|x|$
 c) $x^{(p)} = y^{(p)} \Leftrightarrow x = y$; d) $x^{(p)} = y^{(p+q)} \Leftrightarrow x = y^{(q)}$;
 e) $(\forall x \in F) (\exists!(y, p) \in F \times \mathbb{N}_0)[x = y^{(p)} \ \& \ (\forall z \in F)y \notin z^2]$ ⁶⁾
 f) $x^{[p+1]} = y^{[q+1]} \Rightarrow p = q, x = y$.

Proposition 1.3. *If $G = (G, \cdot) \in \mathcal{V}$, then for each $x \in G$ and $p, q \geq 0$:*

$$(x^{[p]})^2 = (x^{(p)})^2.$$

More generally: $(x^{[p]})^{(r)} = x^{(p+r)}$, for any $r \geq 1$.

Proof. Clearly, the above equality holds for $p = 0$. Suppose that it is true for $p = k$. Then, considering the identity (0.1) and the inductive hypothesis, we have:

$$\begin{aligned} (x^{[k+1]})^2 &= (x^{(k+1)}x^{[k]})^2 = ((x^{(k)})^2x^{[k]})^2 = ((x^{[k]})^2x^{[k]})^2 = ((x^{[k]})^3)^2 = \\ &= ((x^{[k]})^2)^2 = ((x^{(k)})^2)^2 = (x^{(k+1)})^2. \end{aligned}$$

□

Using (1.3), we can define the following infinite set of groupoids:

$$F_1 = \{t \in F : (\forall \alpha \in F) (\alpha^{[1]})^2 \notin P(t)\},$$

$$t, u \in F_1 \Rightarrow t *_{1} u = \begin{cases} tu, & \text{if } tu \in F_1 \\ \alpha^{(2)}, & \text{if } t = u = \alpha^{[1]}. \end{cases}$$

$$F_{n+1} = \{t \in F_n : (\forall \alpha \in F_n) (\alpha^{[n+1]})^2 \notin P(t)\},$$

$$t, u \in F_{n+1} \Rightarrow t *_{n+1} u = \begin{cases} t *_{n} u, & \text{if } t *_{n} u \in F_{n+1} \\ \alpha^{(n+2)}, & \text{if } t = u = \alpha^{[n+1]}. \end{cases}$$

One can show that F_{n+1} is a groupoid and $F_{n+1} \notin \mathcal{V}$.

The fact that $F \supseteq F_1 \supseteq \dots \supseteq F_n \supseteq \dots$ and that F_{n+1} is "better" than F_n suggests to define the carrier of a free groupoid in \mathcal{V} in the following way:

$$R = \{t \in F : (\forall \alpha \in F, k \geq 1) (\alpha^{[k]})^2 \notin P(t)\}. \quad (1.4)$$

(Note that it is not necessary to define the whole sequence, since the desired "good candidate" can be noticed after several steps.)

⁶⁾ \mathbb{N}_0 is the set of nonnegative integers.

By (1.4) we obtain:

$$0) B \subset R \subset F$$

$$i) t, u \in R \Rightarrow \{tu \notin R \Leftrightarrow (\exists \alpha \in F, k \geq 1) t = u = \alpha^{[k]}\}$$

$$ii) t, u \in R \Rightarrow \{tu \in R \Leftrightarrow [t \neq u \text{ or } (t = u \ \& \ (\forall \alpha \in R, k \geq 1) t \neq \alpha^{[k]})]\}$$

$$iii) t^{(p+1)} \in R \Leftrightarrow t \in R \ \& \ t \neq \alpha^{[k]}, k \geq 1.$$

Theorem 1. Let R be defined by (1.4) and an operation $*$ on R by:

$$t, u \in R \Rightarrow \{t * u = tu \text{ if } tu \in R \ \& \ t * u = \alpha^{(k+1)} \text{ if } t = u = \alpha^{[k]}\}.$$

Then $\mathbf{R} = (R, *)$ is a \mathcal{V} -free groupoid with the basis B .

Proof. It follows that, for every $u \in F$, there is at most one pair $(\alpha, k) \in F \times \mathbb{N}_0$, such that $u = \alpha^{[k]}$. By a direct verification of (0.1) we obtain that $R \in \mathcal{V}$. Furthermore, B is a generating set of R and for any groupoid $G \in \mathcal{V}$ and a mapping $\lambda : B \rightarrow G$ there is a homomorphism $\varphi : \mathbf{R} \rightarrow \mathbf{G}$ that extends λ . \square

2. CONSTRUCTION OF \mathcal{V} -FREE OBJECTS IF x^3x^3 IS THE "SUITABLE SIDE"

Now, choose the right-hand side of (0.1) as "suitable" and define:

$$F'_1 = \{t \in F : (\forall \alpha \in F) (\alpha^2)^2 \notin P(t)\}. \quad (2.1)$$

By (2.1) we obtain:

$$1') t, u \in F'_1 \Rightarrow \{tu \notin F'_1 \Leftrightarrow t = u \text{ is a square}\}^7)$$

$$2') t, u \in F'_1 \Rightarrow \{tu \in F'_1 \Leftrightarrow [t \neq u \text{ or } (t = u \text{ is not a square})]\}$$

$$3') t^2 \in F'_1 \Leftrightarrow \{t \in F'_1 \ \& \ t \text{ is not a square}\}$$

$$4') t^2 \in F'_1 \Leftrightarrow t^n \in F'_1, n \geq 3.$$

Define an operation $*'_1$ on F'_1 by:

$$t, u \in F'_1 \Rightarrow t *'_1 u = \begin{cases} tu, & \text{if } tu \in F'_1 \\ (\alpha^3)^2, & \text{if } t = u = \alpha^2. \end{cases}$$

By a direct verification we obtain that $\mathbf{F}'_1 = (F'_1, *'_1)$ is a groupoid. However, the equality

$$(t *'_1 t) *'_1 (t *'_1 t) = ((t *'_1 t) *'_1 t) *'_1 ((t *'_1 t) *'_1 t) \quad (2.2)$$

is not satisfied in \mathbf{F}'_1 . Namely, for $t = \alpha^2$, the left-hand side of (2.2) is $((\alpha^3)^2 \alpha^2)^2$ and the right-hand side is $((\alpha^3)^3)^2$. Thus, $\mathbf{F}'_1 \notin \mathcal{V}$. Therefore, as a consequence of (0.1), we obtain that: $((\alpha^3)^2 \alpha^2)^2 = ((\alpha^3)^3)^2$ is an identity in \mathcal{V} .

This suggests a definition of a new "candidate" $\mathbf{F}'_2 = (F'_2, *'_2)$:

$$F'_2 = \{t \in F'_1 : (\forall \alpha \in F'_1) ((\alpha^3)^2 \alpha^2)^2 \notin P(t)\}$$

⁷ $a \in G$ is a square in a groupoid $\mathbf{G} = (G, \cdot)$ iff $(\exists \alpha \in G) a = \alpha^2$; if G is injective, then α is unique.

$$t, u \in F'_2 \Rightarrow t *'_2 u = \begin{cases} t *'_1 u, & \text{if } tu \in F'_1 \\ ((\alpha^3)^3)^2, & \text{if } t = u = (\alpha^3)^2 \alpha^2. \end{cases}$$

Checking (2.2) (when $*'_1$ is substituted by $*'_2$), we obtain that $F'_2 \notin \mathcal{V}$ and one more identity in \mathcal{V} : $((\alpha^3)^3)^2((\alpha^3)^2 \alpha^2)^2 = ((\alpha^3)^3)^3)^2$.

Continuing this procedure, we can see a "regularity" in the consequences of the identity (2.2). This suggests introducing the following notations:

$$\begin{aligned} x^{<0>} &= x, & x^{<k+1>} &= (x^{<k>})^3; \\ x^{<0|} &= x^2, & x^{<k+1|} &= (x^{<k+1>})^2 x^{<k|} \end{aligned} \quad (2.3)$$

It is easily seen, by induction on n , that:

Proposition 2.1. *If $G = (G, \cdot)$ is any groupoid, then for each $x \in G$ and $m, n \geq 0$:*

$$x^{<m+n>} = (x^{<m>})^{<n>}.$$

By induction on p , one can show the following propositions:

Proposition 2.2. *If $x, y \in F$ and $p, q \geq 0$, then:*

- a) $|x^{<p>}| = 3^p |x|$
- b) $x^{<p>} = y^{<p+q>} \Leftrightarrow x = y^{<q>}$;
- c) $(\forall x \in F)(\exists!(y, p) \in F \times \mathbb{N}_0)[x = y^{<p>} \text{ \& } y \text{ is not a cube}]$.

Proposition 2.3. *If $x, y \in F$ and $p, q \geq 0$, then:*

- a) $|x^{<p|}| = (3^{p+1} - 1)|x|$; b) $|x^{<p|}| < |x^{<p+1|}|$; c) $x^{<p|} \neq x^{<p+m|}$, $m \geq 1$;
- d) $x^{<p+1|} \neq y^{<q|}$, $p \geq 0, q \geq 1$; e) $x^{<p|} = y^{<q|} \Rightarrow p = q, x = y$.

Proposition 2.4. *If $G = (G, \cdot) \in \mathcal{V}$, then for each $x \in G$ and $p, q \geq 0$:*

$$(x^{<p|})^2 = (x^{<p+1|})^2.$$

More generally: $(x^{<p|})^{<r>} = x^{<p+r|}$, for any $r \geq 1$.

As in (1.4), we define the carrier of a free groupoid in \mathcal{V} in the following way:

$$R' = \{t \in F : (\forall \alpha \in F, k \geq 0) (\alpha^{<k|})^2 \notin P(t)\}. \quad (2.4)$$

By (2.4) we obtain:

$$0') \quad B \subset R' \subset F$$

$$i') \quad t, u \in R' \Rightarrow \{tu \notin R \Leftrightarrow (\exists \alpha \in F) t = u = \alpha^{<k|}, k \geq 0\}$$

$$ii') \quad t, u \in R' \Rightarrow \{tu \in R' \Leftrightarrow [t \neq u \text{ or } (t = u \& (\forall \alpha \in F, k \geq 0) t \neq \alpha^{<k|})]\}$$

Theorem 2. *Let R' be defined by (2.4) and an operation $*'$ on R' by:*

$$t, u \in R' \Rightarrow \{t *' u = tu \text{ if } tu \in R' \& t *' u = (\alpha^{<k+1|})^2 \text{ if } t = u = \alpha^{<k|}\}.$$

Then $\mathbf{R}' = (R', *')$ is a \mathcal{V} -free groupoid with the basis B .

Proof. It follows that, for every $u \in F$ there is at most one pair $(\alpha, k) \in F \times \mathbb{N}_0$, such that $u = \alpha^{<k}$. By a direct verification of (0.1) we obtain that $\mathbf{R}' \in \mathcal{V}$. Furthermore, B generates \mathbf{R}' and, for any $\mathbf{G} \in \mathcal{V}$ and a mapping $\lambda : B \rightarrow G$, there is a homomorphism $\varphi : \mathbf{R} \rightarrow \mathbf{G}$ that extends λ . \square

(Note that \mathbf{R} and \mathbf{R}' are isomorphic with the same basis B .)

3. SOME REMARKS

Remark 3.1: The method used above is not applicable in some varieties of groupoids. Namely, consider the variety of groupoids with the identity $x^2 = x^3$. If we choose x^2 as the "suitable side" of the identity and define

$$R = \{t \in F : (\forall \alpha \in F) \alpha^3 \notin P(t)\},$$

$$t, u \in R \Rightarrow \{t * u = tu \text{ if } tu \in R \ \& \ t * u = u^2 \text{ if } t = u^2\},$$

then we obtain that $\mathbf{R} = (R, *)$ is a free object in this variety. However, if we choose the right-hand side as the "suitable" one, then by

$$R' = \{t \in F : (\forall \alpha \in F) \alpha^2 \notin P(t)\},$$

$$t, u \in R' \Rightarrow \{t *' u = tu \text{ if } tu \in R' \ \& \ t *' u = t^3 \text{ if } t = u\},$$

$\mathbf{R}' = (R', *')$ is not a groupoid. (Namely, if $t = u$, then $t *' t = t^3 = t^2 t \notin P(t)$!)

Thus, the procedure used for the variety \mathcal{V} is not applicable in one of the cases for the variety of groupoids with the identity $x^2 = x^3$ and in any variety of groupoids with the identity such that one hand-side of the identity is a part of the other one.

Remark 3.2: It is natural to consider the "shorter" side of the identity $x^2x^2 = x^3x^3$ as a "suitable" one (as we did in Section 1) and to expect a "shorter" (or a "less complicated") construction of a free groupoid in this variety. However, comparing the constructions in Section 1 and Section 2 we can see that they are nearly equal, although one can say that the groupoid powers (1.3) are a "little simpler" than (2.3). Moreover, the situation with the variety defined by $xx^2 = x^2x^2$ is quite different. Namely, the choice of the "shorter" side xx^2 as "suitable" leads to a longer and more complicated construction than the choice of the "larger" side x^2x^2 (the construction in this case finishes at once, at the first step!). (Probably, the "symmetry" in x^2x^2 plays a certain role.)

Remark 3.3: The free groupoids \mathbf{R} and \mathbf{R}' obtained in Theorems 1 and 2 are \mathcal{V} -canonical groupoids. (A groupoid $\mathbf{H} = (H, *)$ is said to be \mathcal{V} -canonical groupoid in a given variety \mathcal{V} ([3]) iff:

$$(c_0) B \subset H \subset F \quad (c_1) tu \in H \Rightarrow t, u \in H \ \& \ tu = t * u; \quad (c_2) \mathbf{H} \text{ is } \mathcal{V}\text{-free}$$

(i.e. $\mathbf{H} \in \mathcal{V}$; B generates \mathbf{H} ; for any $\mathbf{G} \in \mathcal{V}$ and any mapping $\lambda : B \rightarrow G$, there is a homomorphism φ from \mathbf{F} into \mathbf{G} such that $\varphi_B = \lambda$).

For a given variety \mathcal{V} of groupoids, a set R is said to be *representative* for \mathcal{V} ([4]) iff the following conditions are satisfied:

$$(j_0) R \subseteq F;$$

(j_1) for every $w \in F$ there is exactly one $u \in R$ such that $u \in R$ and the equation (w, u) is satisfied in \mathcal{V} ;

(j_2) if $t \in R$, then $P(t) \subseteq R$.

Proposition 3.1. *The carrier of any \mathcal{V} -canonical groupoid is a representative set for \mathcal{V} .*

Proof. Let \mathcal{V} be a variety of groupoids and $\mathbf{R} = (R, *)$ be a \mathcal{V} -canonical groupoid (with a basis B). If \mathbf{F} is an absolutely free groupoid with a basis B , then there is a unique homomorphism φ from \mathbf{F} into \mathbf{R} such that $\varphi(b) = b$, for any $b \in B$. Therefore, for every $w \in F$, $\varphi(w)$ is a uniquely determined element of R and clearly the equation $(w, \varphi(w))$ is satisfied in \mathcal{V} . Thus, (j_1) holds. The condition (j_2) can be shown by induction on length of t . Namely, if $|t| = 1$, i.e. $t \in B$, then $P(t) = \{t\} \subseteq R$. Suppose that $P(t) \subseteq R$ for every $t \in R$ with $|t| \leq k$. Let $t \in R$ be such that $|t| = k+1$. Then $t = uv$, $|u| \leq k$, $|v| \leq k$ and since $\{uv\}$, $P(u)$, $P(v) \subseteq R$, it follows that $P(t) \subseteq R$. \square

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СЛОБОДНИ ГРУПОИДИ СО $x^2x^2 = x^3x^3$

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Резиме

Во работава е даден опис на слободните објекти во многуобразието \mathcal{V} од групоиди дефинирано со идентитетот $x^2x^2 = x^3x^3$. Користена е следнава постапка: едната од двете страни на идентитетот ја сметаме за "соодветна", а другата за "несоодветна". Разгледани се двата можни случаи. Прво, левата страна x^2x^2 е земена за "соодветна". Во тој случај, множеството елементи од F (каде што \mathbf{F} е апсолутно слободен групоид со база B) коишто не содржат дел од обликот x^3x^3 , земено е како "кандидат" за носител на слободен објект во \mathcal{V} . Продолжувајќи ја таа постапка, добиен е \mathcal{V} -слободен групоид. Друг \mathcal{V} -слободен групоид е конструиран со земање на десната страна x^3x^3 како "соодветна". (Добиените \mathcal{V} -слободни групоиди се изоморфни.) Меѓутоа, оваа постапка не е применлива во некои многуобразија групоиди, како на пример во многуобразието дефинирано со идентитетот $x^2 = x^3$, а и во секое многуобразије групоиди со идентитет во кој едната страна е дел од другата

(Remark 3.1). Добиените \mathcal{V} -слободни групоиди \mathbf{R} и \mathbf{R}' (Theorem 1 и Theorem 2) се \mathcal{V} -канонични. Се покажува (Proposition 3.1) дека носителот на \mathcal{V} -каноничен групоид е репрезентативно множество за \mathcal{V} (Remark 3.3)

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