Unique and Minimum Distance Decoding of Linear Codes with Reduced Complexity

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Abstract—We show that for (systematic) linear codes the time complexity of unique decoding is $O\left(n^2q^{nRH(\delta/2/R)}\right)$ and the time complexity of minimum distance decoding is $O\left(n^2q^{nRH(\delta/R)}\right)$. The proposed algorithm inspects all error patterns in the information set of the received message of weight less than d/2 or d, respectively.

Index Terms—nearest neighbor decoding, unique decoding, bounded distance decoding, minimum distance decoding.

I. INTRODUCTION

L ET *C* is a systematic linear code with parameters [n,k,d]. It is well known that Hamming balls Ball(c,t) with radius $t = \lfloor (d-1)/2 \rfloor$ around the codewords $c \in C$ are disjoint. Let *y* is the received message. Then the Unique Decoding strategy is to find the codeword $c_y \in C$, such that $y \in Ball(c_y,t)$, or return incomplete decoding, i.e. $y \notin Ball(c,t) \forall c \in C$. Trivial way to do this is to inspect all q^k codewords and return c_y such that $d(y,c_y) \leq t$. The time complexity of this approach is $O(nq^{Rn})$. Another alternative for unique decoding, with time complexity $O(nq^{H(\delta/2)n})$, is to inspect all V(n,t) error patterns *e* and find the pattern such that y - e is a codeword. Minimum Distance Decoding, on the other hand, inspects all q^k codewords and returns $c_y \in C$ such that $d(y,c_y)$ is minimal or it inspects all error patterns of weight less than the covering radius [1].

In the following section we will show that unique decoding can be done by inspecting all V(k,t) error patterns in the information set of the received message y. Then we will generalize this algorithm to perform Minimum Distance Decoding.

II. THE ALGORITHM

We will use $\langle a | b \rangle$ to denote concatenation of two vectors, such that *a* belongs to the information set and *b* belongs to the check set of a codeword. Let the message *x* is encoded in the codeword $c_x = \langle x | r \rangle$ and sent over a noisy channel. Let the random error pattern is denoted with $e = \langle v | u \rangle$ and the received word is denoted with $y = \langle y_x | y_r \rangle$. The unique decoding algorithm, below, inspects all error patterns in the information set $e = \langle v | 0 \rangle$ with weight $wt(e) \le t$ and outputs the message $\langle x | u \rangle$ if *y* belongs to some Ball(c,t):

Unique_Decoding(y) Let $t = \lfloor (d-1)/2 \rfloor$ 1. find the syndrome $s_y = Hy^T$ 2. if $wt(s_y) \le t$ return y 3. foreach vector $e = \langle v | 0 \rangle$ s.t. $wt(e) \le t$ a. find $s_e = He^T$ b. if $wt(e) + wt(s_y - s_e) \le t$ return y-e 4. return -1 // incomplete decoding

Proposition 1: The Unique_Decoding(y) algorithm can remove any error pattern of weight $\leq t$ from the received message y.

Proof: Let $e_v = \langle v | 0 \rangle$, $wt(e_v) \leq t$, is the coset leader and $s_v = He_v^T$ is the syndrome of a coset. Let assume that the pairs (s_v, e_v) are explicitly known; for example, they are stored in a look-up table.

We will consider the error pattern $e = \langle v | u \rangle$ as a linear combination of two vectors

$$e = \langle v | 0 \rangle + \langle 0 | u \rangle = e_v + e_u \tag{1}$$

Since $wt(e_u) \le t$ and $s_u = He_u^T = u$, we can say that e_u is

the leader and u is the syndrome of the same coset. Hence, the syndrome s of the received message y is

$$s = Hy^{T} = H(c_{x} + e_{y} + e_{u})^{T} = s_{y} + u$$
 (2)

From (2), we can formulate the decoding strategy: for each e_v in the table (s_v, e_v) denote with $x = y - e_v$ and compute the syndrome

$$s_x = Hx^T \tag{3}$$

If

$$wt(e_v) + wt(s_x) \le t \tag{4}$$

then the error pattern that corrupted the message is

$$e = \left\langle e_{v} \left| s_{x} \right\rangle \right.$$

In worst case, the algorithm will check all $V_q(k,t)$ error patterns. So the time complexity is upper-bounded by

$$O\left(n^2 q^{RH\left(\frac{\delta}{2R}\right)n}\right)$$

If we use the fact that for long random linear codes the covering radius is equal to d, where d is the largest integer solution of the Gilber-Varshamov inequality [2]. Then we can formulate Minimum Distance Decoding algorithm that inspects all error patterns of weight less than d in the information set:

 $MD_Decoding(y)$ $error_vector=0, error_wt=n$ 1. compute the syndrome $s_y=Hy^T$ 3. foreach vector $e=\langle v | 0 \rangle$, wt(e) $\leq d$ a. Find $s_e=He^T$ b. if wt(e)+wt(s_y-s_e) \leq error_wt i. error_wt=wt(e)+wt(s_y-s_e) ii. error_vector=e 4. return y-error vector

The proof of correctness of the above algorithm is similar to the proof of proposition 1. Thus the time complexity of MDD decoding is

$$O\left(n^2 q^{RH\left(\frac{\delta}{R}\right)n}\right)$$

and this result improves some of the previously known bounds on MD decoding found in [1], [2], [3], or [4].

On the other hand, pairs (s_v, e_v) need not to be stored in a look-up table, but they can be listed by divide-and-conquer strategy in the course of decoding. Therefore, the space complexity of $Unique_Decoding(y)$ and $MD_Decoding(y)$ is proportional with the dimension of the generator matrix, i.e. $O(n^2)$.

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