International Journal of Algebra, Vol. 2, 2008, no. 10, 451 - 461

Free Groupoids in the Class of Power Left and Right Idempotent Groupoids

Vesna Celakoska-Jordanova

Faculty of Natural Sciences and Mathematics
"Ss. Cyril and Methodius University"
P.O.Box 162, Skopje, Macedonia
vesnacj@iunona.pmf.ukim.edu.mk

Abstract

The subject of this paper is the class of groupoids such that any groupoid $\boldsymbol{G} = (G, \cdot)$ of this class has the property: every subgroupoid of $\boldsymbol{G}$ generated by any element $a \in G$ satisfies the identity $(xx)(yy) \approx xy$. It is shown that this class is a variety. A construction and a characterization of free groupoids in this variety are obtained. The word problem is solvable for this variety.

Mathematics Subject Classification: Primary 08B20; Secondary 03C05

Keywords: groupoid, subgroupoid, groupoid power, variety of groupoids, free groupoid

We will work with groupoids, i.e. algebras with one binary operation. In [5] the variety $\mathcal{V}$ of left and right idempotent (LRI) groupoids, i.e. groupoids that satisfy the identity $(xx)(yy) \approx xy$, is considered. We investigate a class of groupoids larger than $\mathcal{V}$, called the class of power left and right idempotent (PLRI) groupods. A groupoid $\boldsymbol{G} = (G, \cdot)$ belongs to the class of PLRI groupoids if and only if for each element $a \in G$ the subgroupoid generated by a belongs to $\mathcal{V}$. We denote this class by $p\mathcal{V}$. Throughout the paper we will use the notion of groupoid power and some of its properties, stated in [6] and [8].

The purpose of this paper is to obtain: an axiom system for the class $p\mathcal{V}$, a description of free groupoids in $p\mathcal{V}$ and their characterization, using a larger class of groupoids, called the class of injective groupoids in $p\mathcal{V}$. It is shown in [5] that the word problem is solvable for the variety of LRI groupoids. The word problem is also solvable for the variety of PLRI groupoids. The main

[1]Research support by Macedonian Academy of Sciences and Arts, within the project "Algebraic structures" (project manager G.Čupona)

results of this paper are Theorems 2.1, 2.4, 3.3 and 4.4.

For the notation and basic notions of universal algebra the reader is referred to [4]. In most cases, without mention, the operation is denoted multiplicatively: the product of two elements x, y of a groupoid is denoted by $x \cdot y$ or just xy. If $x = y$, we put x^2 instead of xx and call it the square of x. Hence, $x^2y^2 = (xx)(yy)$, $x^2y = (xx)y$, $xy^2 = x(yy)$ etc.

1 Preliminary Notes

In what follows, $B \neq \emptyset$ will be an arbitrary set, and T_B will denote the set of all groupoid terms over B in the signature $\cdot$. The terms are denoted by $t, u, v, \dots, x, y, \dots$ $\boldsymbol{T}_B = (T_B, \cdot)$ is the absolutely free groupoid (i.e. free in the class of all groupoids) with the free basis B, where the operation is defined by $(u, v) \mapsto uv$. For any term $v \in T_B$ we define the *length* $|\,v\,|$ of v and the *set of subterms* $P(v)$ of v by:

$$|\,b\,| = 1,\ |\,tu\,| = |\,t\,| + |\,u\,|;\ P(b) = \{b\},\ P(tu) = \{tu\} \cup P(t) \cup P(u),$$

for any $b \in B$ and any $t, u \in T_B$.

Let $\boldsymbol{G} = (G, \cdot)$ be any groupoid and A a nonempty subset of G. If $\boldsymbol{Q}$ is the subgroupoid of $\boldsymbol{G}$ generated by A, then $Q = \cup\{A_k : k \geq 0\}$, where $A_0 = A$, $A_{k+1} = A_k \cup A_kA_k$. Define a mapping $\chi_Q : Q \to N_0$, where N_0 is the set of nonnegative integers, by $\chi_Q(x) = \min\{k \in N_0 : x \in A_k\}$.

Specially, $\boldsymbol{T}_B = \cup\{B_k : k \geq 0\}$, where $B_0 = B$, $B_{k+1} = B_k \cup B_kB_k$. The hierarchy $\chi : T_B \to N_0$ has the property $\chi(tu) = 1 + \max\{\chi(t), \chi(u)\}$, for all $t, u \in T_B$. Note that this is not true for an arbitrary groupoid $\boldsymbol{G}$.

The groupoid $\boldsymbol{T}_B$ is *injective*, i.e. the operation $\cdot$ in $\boldsymbol{T}_B$ is an injective mapping: $xy = vw \Rightarrow x = v, y = w$. The set B is the set of primes in $\boldsymbol{T}_B$ that generates $\boldsymbol{T}_B$. (An element a of a groupoid $\boldsymbol{G} = (G, \cdot)$ is said to be *prime* in $\boldsymbol{G}$ if and only if $a \neq xy$, for all $x, y \in G$.) This two properties of T_B characterize all absolutely free groupoids ([1]; Lemma 1.5).

Theorem 1.1 *A groupoid $\boldsymbol{H} = (H, \cdot)$ is an absolutely free groupoid if and only if it satisfies the following two conditions:*

(i) $\boldsymbol{H}$ is injective;

(ii) The set of primes in $\boldsymbol{H}$ is nonempty and generates $\boldsymbol{H}$.

Then the set of primes is the unique free basis of $\boldsymbol{H}$.

We refer to this proposition as *Bruck Theorem* for the class of all groupoids.

By $\boldsymbol{T}_e = (T_e, \cdot)$ we denote the absolutely free groupoid with one-element basis $\{e\}$. The terms over $\{e\}$ are called *groupoid powers* ([5]) and will be denoted by $f, g, h, \dots$. We will state a few properties of groupoid powers that will be used further on.

For any groupoid $\boldsymbol{G} = (G, \cdot)$, each term f over $\{e\}$ induces a transformation $f^G : G \to G$, called an *interpretation* of f in $\boldsymbol{G}$, defined by $f^G(x) = \varphi_x(f)$, where $\varphi_x : T_e \to G$ is the homomorphism from $\boldsymbol{T}_e$ into $\boldsymbol{G}$ such that $\varphi_x(e) = x$. In other words, for any $g, h \in T_e$ and $x \in G$

$$e^G(x) = x, \quad (gh)^G(x) = g^G(x)h^G(x).$$

We write $f(x)$ instead of $f^G(x)$ when $\boldsymbol{G}$ is understood.

Next statements concerning groupoid powers are shown in [8].

Theorem 1.2 *If $f, g \in T_e$, $t, u \in T_B$, then*
a) $|\, f(t)\,| = |\, f\,| \cdot |\, t\,|$;
b) $f(t) = g(u) \;\;\&\;\; (|\, t\,| = |\, u\,| \;\vee\; |\, f\,| = |\, g\,|) \;\Leftrightarrow\; (f = g \;\;\&\;\; t = u)$;
c) $f(t) = g(u) \;\;\&\;\; |\, t\,| \geq |\, u\,| \;\Leftrightarrow (\exists!\, h \in T_e)\, (t = h(u) \;\;\&\;\; g = f(h))$.

If an operation "$\circ$" is defined on the set T_e by: $f \circ g = f(g)$, then one obtains that $(T_e, \circ, e)$ is a cancellative monoid ([6], Prop. 1.6).

An element $c \in G$ is said to be *primitive* in $\boldsymbol{G} = (G, \cdot)$ if and only if for any $a \in G$ and any $f \in T_e$, the following implication is true: $c = f(a) \Rightarrow f = e$.

Specially, primitive elements in $\boldsymbol{T}_B$ are called *primitive terms*.

We essentially use Lemma 1.3 ([2], Lemma 3.1) in the proof of Lemma 1.4 and in the next section.

Lemma 1.3 *For any term $v \in T_B$ there is a uniquely determined primitive term $u \in T_B$ and a uniquely determined groupoid power $f \in T_e$ such that $v = f(u)$.*

In that case we call u the *base* of v, f the *power* of v, $|\, f\,|$ the *exponent* of v, and denote by $\underline{v}$, $v^{\sim}$, $|\, v^{\sim}\,|$, respectively.

Lemma 1.4 *Let $v, w \in T_B$.*
a) If v, w have different bases, then vw is a primitive term in T_B.
b) The terms v, w have the same base t if and only if t is the base of vw and $(vw)^{\sim} = v^{\sim}w^{\sim}$.

Proof. Let $v, w \in T_B$. Then, by Lemma 1.3 $v = f(t)$, $w = g(u)$ and $vw = h(\alpha)$, where $t = \underline{v}$, $u = \underline{w}$, $\alpha = \underline{vw}$, $f = v^{\sim}$, $g = w^{\sim}$, $h = (vw)^{\sim}$.

a) Let $t = \underline{v} \neq \underline{w} = u$. We will show that $h = e$. Suppose that $h \neq e$. Then, there are $h_1, h_2 \in T_e$, such that $h = h_1h_2$. The following is true:

$$f(t)g(u) = vw = h(\alpha) = (h_1h_2)(\alpha) = (h_1(\alpha))(h_2(\alpha)).$$

Since $\boldsymbol{T}_B$ is injective, it follows that $h_1(\alpha) = f(t)$ and $h_2(\alpha) = g(u)$. If $|\,\alpha\,| = |\, t\,|$, then by Theorem 1.2 b), $\alpha = t$, $h_1 = f$. Therefore, $h_2(t) = g(u)$. There are three cases (only): $|\, t\,| = |\, u\,|$, $|\, t\,| > |\, u\,|$ and $|\, u\,| > |\, t\,|$. The case

$|t| = |u|$ is not possible. Namely, by Theorem 1.2 b), it follows that $t = \alpha = u$, that contradicts the supposition $t \neq u$. The case $|t| > |u|$ is not possible, too. Indeed, if $h_2(t) = g(u)$, then by Theorem 1.2 c), there is a unique $l \in T_e$ such that $t = l(u)$, $g = h_2(l)$. As u is a primitive term, it follows that $l = e$, i.e. $t = e(u) = u$, that contradicts $t \neq u$. By the same argument, the case $|u| > |t|$ is not possible, too.

b) Let $\underline{v} = \underline{w} = t$. Then $h(\alpha) = f(t)g(t) = (fg)(t)$. Since α is a primitive term and $|fg| \geq 2$, it follows that $h \neq e$. Let $h = h_1h_2$. Then $f(t)g(t) = h(\alpha) = (h_1h_2)(\alpha) = (h_1(\alpha))(h_2(\alpha))$. Since $\boldsymbol{T}_B$ is injective, it follows that $h_1(\alpha) = f(t)$, $h_2(\alpha) = g(t)$. As t and α are primitive terms, neither $|t| > |\alpha|$ nor $|\alpha| > |t|$. Therefore, $|t| = |\alpha|$. By Theorem 1.2 b), it follows that $f = h_1$, $g = h_2$ and $t = \alpha$. Thus t is the base of vw. By $vw = (fg)(t)$, it follows that $(vw)^{\sim} = h = fg = v^{\sim}w^{\sim}$.

Conversely, let $t = \underline{v} \neq \underline{w} = u$. Then by a), vw is the base of vw, so $h = e$. This implies that $h \neq fg$, that contradicts $(vw)^{\sim} = v^{\sim}w^{\sim}$.

2 Construction of free groupoids in the class of PLRI groupoids

Here, the class of LRI groupoids, i.e. groupoids that satisfy the identity $x^2y^2 \approx xy$ is denoted by $\mathcal{V}$. It is shown in [5] that the axiom $x^2y^2 \approx xy$ is equivalent with the system of axioms $x^2y \approx xy$, $xy^2 \approx xy$ and that, if $\boldsymbol{G} \in \mathcal{V}$, then $a \in G$ is a square if and only if a is an idempotent.

A groupoid $\boldsymbol{G} = (G, \cdot)$ is said to be power left and right idempotent (PLRI) if and only if for any $a \in G$ the subgroupoid of $\boldsymbol{G}$ generated by $\{a\}$, denoted by $\langle a \rangle$, is LRI groupoid. The class of all PLRI groupoids is denoted by $p\mathcal{V}$.

The following theorem shows that the elements of any groupoid of the class $p\mathcal{V}$ have almost trivial powers, i.e. if $f(x)$ is a power of x, then either $f(x) = x$ or $f(x) = x^2$, for any groupoid power f, $f \neq e$.

Theorem 2.1 *$\boldsymbol{G} \in p\mathcal{V}$ if and only if $(\forall x \in G)(\forall f \in T_e \setminus \{e\})\ f(x) = x^2$.*

Proof. The direct statement can be shown by induction on length $|f|$. Conversely, let $y, z \in \langle x \rangle$. Then there are $f, g \in T_e$ such that $y = f(x)$, $z = g(x)$. In that case, $yz = f(x)g(x) = (fg)(x)$ and $y^2z^2 = (f(x))^2(g(x))^2 = (f^2g^2)(x)$. Let $h = fg$, $l = f^2g^2$. Since $|h| = |fg| \geq 2$ and $|l| = |f^2g^2| \geq 4$, it follows that $h, l \in T_e \setminus \{e\}$. By the hypothesis $f(x) = x^2$ for all $f \in T_e \setminus \{e\}$, we obtain that $h(x) = x^2 = l(x)$. Therefore, $y^2z^2 = (f^2g^2)(x) = x^2 = h(x) = l(x) = (fg)(x) = yz$ and thus $\langle x \rangle \in \mathcal{V}$.

As a consequence, we obtain:

Corollary 2.2 *A groupoid $\boldsymbol{G} = (G, \cdot)$ is a $p\mathcal{V}$-groupoid if and only if*

$$(\forall x \in G)(\forall f, g \in T_e)\ (f(x))^2(g(x))^2 = f(x)g(x).$$

Theorem 2.1 enables to obtain a finite system of axioms for the variety $p\mathcal{V}$ (and thus the variety $p\mathcal{V}$ is finitely based).

Corollary 2.3 *The class of $p\mathcal{V}$-groupoids is a variety defined by the identities $x^2 \approx x^2x \approx xx^2 \approx x^2x^2$.*

Now, we are ready to start a construction of free groupoids in $p\mathcal{V}$. Assuming that B is a nonempty set and $\boldsymbol{T}_B = (T_B, \cdot)$ the absolutely free groupoid with the free basis B, we are looking for a *canonical groupoid* ([7]) in $p\mathcal{V}$, i.e. a groupoid $\boldsymbol{R} = (R, *)$ with the following properties:

i) $B \subset R \subset T_B$, *ii)* $tu \in R \Rightarrow t, u \in R \ \& \ t * u = tu$

iii) $\boldsymbol{R}$ is a free groupoid in $p\mathcal{V}$ with the free basis B.

Define the carrier of the desired groupoid $\boldsymbol{R}$ as the set

$$R = \{t \in T_B : (\forall u \in P(t))\ |\, u^{\sim} \,| \leq 2\}. \tag{1}$$

Thus, $t \in R$ if and only if t has no subterm that has an exponent greater than 2. The following properties of R are clear:

a) $B \subset R \subset T_B$; $t \in R \Rightarrow P(t) \subset R$.

b) $t, u \in R \ \Rightarrow [tu \notin R \Leftrightarrow (\exists x \in R)(\exists f \in T_e, |\, f \,| \geq 3)\ tu = f(x)]$.

(Namely, since $t, u \in R$, it follows that: $t = f_1(x)$, $u = f_2(x)$, where $|f_1|, |f_2| \leq 2$ and $x = \underline{t} = \underline{u}$. If $tu \notin R$, then $|f_1| = 2$ or $|f_2| = 2$. Thus $|\, f \,| = |\, f_1 f_2 \,| = |\, f_1 \,| + |\, f_2 \,| \geq 3$. The converse is obvious.)

c) $t, u \in T_B \Rightarrow [tu \in R \Leftrightarrow t, u \in R \ \& \ |\,(tu)^{\sim}\,| = 2]$.

Define an operation $*$ on R by:

$$t, u \in R \Rightarrow t * u = \begin{cases} tu, & \text{if } tu \in R \\ x^2, & \text{if } \underline{t} = \underline{u} = x, |\, t^{\sim} \,| + |\, u^{\sim} \,| \geq 3 \end{cases} \tag{2}$$

One can show that $\boldsymbol{R} = (R, *)$ is a groupoid. By (2), Theorem 2.1 and induction on length, one can show the properties $1^\circ - 4^\circ$.

1°. B is the set of primes in $\boldsymbol{R}$ and generates $\boldsymbol{R}$.

The interpretation of groupoid powers in the groupoid $\boldsymbol{R}$ is given by:

$$e_*(t) = t,\ (fg)_*(t) = f_*(t) * g_*(t),$$

for any $t \in R$ and $f, g \in T_e$.

2°. Let $t \in R$ and $f \in T_e \setminus \{e\}$.

a) If t is not a square in $\boldsymbol{T}_B$, then $f_*(t) = t^2$, and, specially, $t * t = t^2$.

b) If t is a square of x in $\boldsymbol{T}_B$, where x is a primitive term in $\boldsymbol{T}_B$, then

$f_*(t) = t$, and, specially, $t * t = t$.

As a consequence of 2° one obtains that for each $f \in T_e \setminus \{e\}$ and $t \in R$, $f_*(t) = t * t$, and therefore (by Theorem 2.1):

3°. $\boldsymbol{R} \in p\,\mathcal{V}$.

4°. For any $\boldsymbol{G} \in p\,\mathcal{V}$ and any mapping $\lambda : B \to G$, there is a homomorphism $\psi : \boldsymbol{R} \to \boldsymbol{G}$ that extends λ, i.e. $\boldsymbol{R}$ has the universal mapping property for $p\,\mathcal{V}$ over B.

Proof. Let φ be the homomorphism from $\boldsymbol{T}_B$ into $\boldsymbol{G}$ that extends λ. Let $\psi : R \to G$ be the restriction of φ on R, i.e. $\psi = \varphi|_R$. It suffices to show that for any $t, u \in R$, $\varphi(t * u) = \varphi(t)\varphi(u)$. If $tu \in R$, then $t * u = tu$, and the proposition clearly holds. Let $tu \notin R$, i.e. let $t = f(x)$, $u = g(x)$, where $x = \underline{t} = \underline{u}$ and $|\,f\,| \le 2$, $|\,g\,| \le 2$, $|\,fg\,| \ge 3$. Then $t * u = x^2$, so $\varphi(t * u) = \varphi(x^2) = \varphi((fg)(x)) = \varphi(f(x)g(x)) = \varphi(tu) = \varphi(t)\varphi(u)$.

By 1° to 4° it follows that:

Theorem 2.4 *The groupoid $\boldsymbol{R} = (R, *)$, defined by* (1) *and* (2) *is free in $p\mathcal{V}$ with the free basis B.*

Note that, if $a, b \in B$ are distinct elements in B, then $a * (b * b) = a * b^2 = ab^2 \ne ab = a * b$, and therefore: if $|\,B\,| \ge 2$, then $\boldsymbol{R} \notin \mathcal{V}$.

Since $\boldsymbol{R}$ has the properties: $B \subset R \subset T_B$; $tu \in R \Rightarrow t, u \in R \;\&\; t * u = tu$; $\boldsymbol{R}$ is free in $p\mathcal{V}$ with the free basis B, it follows that $\boldsymbol{R}$ is a canonical groupoid in $p\mathcal{V}$. The class of free groupoids in $p\mathcal{V}$ will be denoted by $p\mathcal{V}_f$.

The following properties of $(R, *)$ are clear.

Lemma 2.5 *a) For any $x \in R$, $x * x$ is an idempotent in $\boldsymbol{R}$.*
b) $t \in R$ is an idempotent in $\boldsymbol{R}$ if and only if t is a square in $\boldsymbol{R}$.
c) If $t \in R$ is an idempotent in $\boldsymbol{R}$, then there is a unique nonidempotent $x \in R$ (i.e. $x \ne x * x$) such that $t = x * x$.
d) $t \in R$ is primitive in $\boldsymbol{R}$ if and only if t is primitive in $\boldsymbol{T}_B$.

Lemma 2.6 *Let I be the set of all idempotents in $\boldsymbol{R}$ and N be the set of all nonidempotents in $\boldsymbol{R}$. If $z \in R$ is a nonidempotent and z is not prime in $\boldsymbol{R}$, then one of the following cases is possible:*
1) $z = \alpha * \beta$, $\alpha, \beta \in I$, $\alpha \ne \beta$; *3) $z = \alpha * x$, $\alpha \in I$, $x \in N$;*
2) $z = x * \alpha$, $\alpha \in I$, $x \in N$; *4) $z = x * y$, $x, y \in N$, $x \ne y$.*

Lemma 2.7 *For any $t \in R \setminus B$ there is exactly one pair $(u, v) \in R^2$, such that $t = uv = u * v$.*

We say that (u, v) is the *pair of divisors* of t in $\boldsymbol{R}$. In this case: $u = v$ if and only if $u^2 \in R$ (if and only if u is not a square); then we say that u is the *divisor* of t.

Remark. The Lemma 2.7 does not exclude existence of distinct pairs (u_1, v_1), (u_2, v_2) in R^2, such that $u_1 * v_1 = u_2 * v_2$. For example, if $a \in B$, then $(a, a^2), (a^2, a^2) \in R^2$ and $a * a^2 = a^2 = a^2 * a^2$. Note that aa^2 and a^2a^2 are not in R.

3 Injective groupoids in the variety of PLRI groupoids

In this section we define a subclass of the class $p\mathcal{V}$ larger than the class $p\mathcal{V}_f$, called the class of injective groupoids in $p\mathcal{V}$. For defining the class of injective groupoids as a subclass of a given variety $\mathcal{W}$ we essentially use properties of the corresponding canonical groupoid in $\mathcal{W}$, that has to be previously constructed. We will look for an axiom system of the class of injective groupoids in $\mathcal{W}$ among the properties of the corresponding canonical groupoid $\boldsymbol{R} = (R, *)$ in $\mathcal{W}$, that are related to the properties of the elements in $\boldsymbol{R}$ that are not $\mathcal{W}$-prime. If the identities that are the axioms of the variety $\mathcal{W}$ are normal, i.e. neither of its sides is a variable, then $\mathcal{W}$-prime element in $\boldsymbol{R}$ means the same as prime element defined before Theorem 1.1. Here, we will use the suitable properties of $(R, *)$ contained in Lemmas 2.5, 2.6, 2.7.

We say that a groupoid $\boldsymbol{H} = (H, \cdot)$ is *injective* in the variety $p\mathcal{V}$ if and only if the following conditions are satisfied:

0) $\boldsymbol{H} \in p\mathcal{V}$

1) If $a \in H$ is an idempotent, then there is a unique nonidempotent $c \in H$, such that $a = c^2$ and the equality $a = xy$ holds if and only if $\{x, y\} \subseteq \{c, c^2\}$. (In that case we say that c is the *divisor* of a or c is the *base* of a.)

2) If $a \in H$ is a nonidempotent and is not prime in $\boldsymbol{H}$, then there is a unique pair $(c, d) \in H^2$, such that $a = cd$ and $\underline{c} \neq \underline{d}$. (Note that c, d could be both idempotents; one idempotent and the other nonidempotent; both nonidempotents.)

The class of all injective groupoids in $p\mathcal{V}$ is denoted by $p\mathcal{V}_i$. From the definition of injective groupoid in $p\mathcal{V}$ and Lemmas 2.5, 2.6 and 2.7 we obtain:

Corollary 3.1 *Every free groupoid in $p\mathcal{V}$ is injective in $p\mathcal{V}$.*

If $\boldsymbol{H} = (H, \cdot) \in p\mathcal{V}_i$ and $a \in H$, then the subgroupoid $Q = \{a^2\}$ of $\boldsymbol{H}$ is not injective in $p\mathcal{V}$. This implies the following proposition:

Corollary 3.2 *Neither of the classes $p\mathcal{V}_f$, $p\mathcal{V}_i$ is hereditary.*

The following statement gives a description of free groupoids in $p\mathcal{V}$ within the class of injective groupoids in $p\mathcal{V}$.

Theorem 3.3 (Bruck Theorem for the variety $p\mathcal{V}$) *A groupoid $\boldsymbol{H} \in p\mathcal{V}$ is free in $p\mathcal{V}$ if and only if the following conditions are satisfied:*
(i) $\boldsymbol{H}$ is injective in $p\mathcal{V}$.
(ii) The set of primes in $\boldsymbol{H}$ is nonempty and generates $\boldsymbol{H}$.

Proof. The "if part" follows by Corollary 3.1 and Theorem 2.4. Conversely, let $\boldsymbol{H} \in p\mathcal{V}$ and (i) and (ii) be satisfied. Define a sequence of sets $(H_k : k \geq 0)$ by $H_0 = P$, $H_{k+1} = H_k \cup H_k H_k$, where P is the set of primes in $\boldsymbol{H}$. Then $H = \cup\{H_k : k \geq 0\}$. Define a hierarchy of the elements in $\boldsymbol{H}$ as a mapping $\chi : H \to N_0$ by:

$$\chi(a) = \begin{cases} 0, & \text{if } a \in H_0 \\ k+1, & \text{if } a \in H_{k+1} \setminus H_k. \end{cases}$$

Note that, if $P = \{a\} = H_0$, then $H_1 = \{a, a^2\} = H_2 = \ldots$, i.e. for any $k \geq 1$, $H_{k+1} \setminus H_k = \emptyset$. However, if $|\,P\,| \geq 2$, then the sequence $(H_k : k \geq 0)$ will not terminate, i.e. $H_{k+1} \setminus H_k \neq \emptyset$, for each $k \geq 0$. Namely, $H_0 = P = \{a, b, \ldots\}$, $a \neq b$, implies that $H_1 = H_0 \cup \{a^2, b^2, ab, \ldots\}$. There are at least two idempotents in H_1 with distinct bases, for instance a^2 and b^2, and at least two nonprime nonidempotents, for instance ab and ba in H_1, that are not in H_0. Thus $H_1 \setminus H_0 \neq \emptyset$. There are at least two idempotents with distinct bases in H_2, for instance $(ab)^2$ and $(ba)^2$, that are not in H_1 and also, there are at least two nonidempotents in H_2, for instance $(ab)(ba)$ and $a^2(ab)$, that are not in H_1. Thus $H_2 \setminus H_1 \neq \emptyset$. Continuing this way, we come to the conclusion that $H_{k+1} \setminus H_k \neq \emptyset$, for each $k \geq 0$. (We come to the same conclusion if we put: $c_0 = ab$, $c_1 = c_0 a$, $c_2 = c_1 b, \ldots$, $c_{2k+1} = c_{2k} a$, $c_{2k+2} = c_{2k+1} b$, where $k \geq 0$. Then, for each nonnegative integer n, $c_n \in H_n \setminus H_{n-1}$, i.e. $H_n \setminus H_{n-1} \neq \emptyset$.)

For any product cd in $\boldsymbol{H}$ the following equality is true:

$$\chi(cd) = 1 + \max\{\chi(c), \chi(d)\}. \tag{3}$$

We will prove this equality by induction on hierarchy. If $a \in H_1 \setminus H_0$, then $a = cd$, where $c, d \in H_0$, $\chi(c) = \chi(d) = 0$ and $\chi(a) = 1$. If $a = cd$ is an idempotent, then $c = d$, i.e. $a = c^2$ and thus $1 + \max\{\chi(c), \chi(c)\} = 1 = \chi(a) = \chi(cd)$. If $a = cd$ is a nonidempotent (clearly $c \neq d$), then $1 + \max\{\chi(c), \chi(d)\} = 1 + 0 = 1 = \chi(a) = \chi(cd)$. Thus, (3) holds for $k = 0$. Suppose that (3) is true for $k \leq n$ and let $cd \in H_{n+1} \setminus H_n$. Then $c, d \in H_n$, $\max\{\chi(c), \chi(d)\} = n$ and $\chi(cd) = n + 1$. Therefore, $1 + \max\{\chi(c), \chi(d)\} = n + 1 = \chi(cd)$. Hence, (3) is true, for each $k \geq 0$.

Now, we are ready to prove the "only if part" of Theorem 3.3, i.e. that $\boldsymbol{H}$ is free in $p\mathcal{V}$ when (i) and (ii) are satisfied. Let $G \in p\mathcal{V}$ and $\lambda : P \to G$ be any mapping. Define a sequence of mappings $\psi_0, \psi_1, \ldots, \psi_k, \ldots$ on H_0, $H_1 = H_0 \cup H_0 H_0$, $\ldots$, $H_k = H_{k-1} \cup H_{k-1} H_{k-1}$, $\ldots$, respectively, as follows:

$\psi_0 : H_0 \to G$, by $\psi_0 = \lambda$; $\psi_1 : H_1 \to G$, by $\psi_1(a) = \psi_0(a)$, if $a \in H_0$ and $\psi_1(a) = \psi_0(c)\psi_0(d)$, if $a = cd$, where $c, d \in H_0$.

Suppose that ψ_i is defined for all $1 \leq i \leq k$.

Define a mapping $\psi_{k+1} : H_{k+1} \to G$ by:

$$\psi_{k+1}(a) = \begin{cases} \psi_k(a), \text{ if } a \in H_k, \\ \psi_k(c)\psi_k(d), \text{ if } a \in H_{k+1} \setminus H_k, (c,d) \text{ is the pair of divisors of } a; \\ (\psi_k(c))^2, \text{ if } a \in H_{k+1} \setminus H_k, a = c^2, \text{ i.e. c is the divisor of } a. \end{cases}$$

The homomorphism condition for ψ_i is satisfied for the "bad products" too. Indeed, $\psi_i(xx^2) = \psi_i(x^2) = \psi_i(x)\psi_i(x) = \psi_i(x)(\psi_i(x))^2$.

So, we obtain an infinite sequence of mappings $\psi_0, \psi_1, \ldots, \psi_k, \ldots$ such that every ψ_{i+1} is an extension of ψ_i, $i \geq 0$. Putting $\psi = \cup\{\psi_i : i \geq 0\}$ we obtain that $\psi : H \to G$ is a well-defined mapping and homomorphism from $\boldsymbol{H}$ into $\boldsymbol{G}$, i.e. for any $xy \in H$, $\psi(xy) = \psi(x)\psi(y)$. Namely, $xy \in H$ implies that there is a positive integer r such that $xy \in H_r$, and thus $\psi(xy) = \psi_r(xy) = \psi_r(x)\psi_r(y) = \psi(x)\psi(y)$.

Thus, the universal mapping property for $p\mathcal{V}$ over P is satisfied. Hence, $\boldsymbol{H}$ is a free groupoid in $p\mathcal{V}$ with the free basis P.

Bellow a *construction of injective groupoid in* $p\mathcal{V}$ *that is not free in* $p\mathcal{V}$ is given.

Let N be an infinite set, I a set equivalent and disjoint with N, $H = N \cup I$ and $\varphi : N \to I$ a bijection. Let $D = \{(x,y) : x, y \in N \cup I, x \neq y\}$. Since N is infinite it follows that the sets N, I and D are equivalent. Therefore, there is an injective mapping $\psi : D \to N$. Define an operation "$\cdot$" in $H = N \cup I$ by:

$$n \cdot n = \varphi(n), \quad i \cdot i = i, \quad x \cdot y = \psi(x,y),$$

for any $n \in N$, $i \in I$, $(x,y) \in D$.

One can verify that $\boldsymbol{H} = (H, \cdot)$ is a groupoid that is injective in $p\mathcal{V}$. If ψ is a bijection, then there are no prime elements in $\boldsymbol{H}$. So, by the Bruck Theorem for the variety $p\mathcal{V}$ (Theorem 3.3), $\boldsymbol{H}$ is not a free groupoid in $p\mathcal{V}$. Therefore, the following proposition holds.

Theorem 3.4 *The class of free groupoids in* $p\mathcal{V}$ *is a proper subclass of the class of injective groupoids in* $p\mathcal{V}$, *i.e.* $p\mathcal{V}_f \subset p\mathcal{V}_i$.

4 Word problem for the variety of PLRI groupoids

We will use an Evans's result ([3]) to show that the word problem is solvable for the variety $p\mathcal{V}$.

For a groupoid $\boldsymbol{G}$ and a nonempty subset D of G we define a *partial groupoid* D_G with the underlying set D as follows: for $a, b \in D$ the product ab is defined in D_G if and only if this product in $\boldsymbol{G}$ belongs to D, and in this case the product in D_G is equal to the product in $\boldsymbol{G}$.

By a homomorphism of a partial groupoid A into a partial groupoid B we mean a mapping $f : A \to B$ such that: whenever a, b are elements of A such that ab is defined, then $f(a)f(b)$ is defined in B and $f(ab) = f(a)f(b)$. We say that A is *embeddable* into B if there is an injective homomorphism of A into B. We say that a partial groupoid A is *strongly embeddable* into a groupoid $\boldsymbol{G}$ if it is isomorphic to D_G, for a nonempty subset D of G.

Clearly, if a partial groupoid A is strongly embeddable into a $p\mathcal{V}$-groupoid, then it satisfies the following condition:

(0) if $a \in A$ is such that a^2 is defined, then a^2a, aa^2 and a^2a^2 are also defined and $a^2a = aa^2 = a^2a^2 = a^2$.

For a partial groupoid A satisfying (0) we define a groupoid $(G, \circ)$ as follows:

(i) if xy is defined in A, then $x \circ y = xy$

(ii) if x^2 is not defined in A, then $x \circ x = x$

(iii) if xy is not defined in A and $x \neq y$, then $x \circ y = c$, where c is a fixed element in A.

Theorem 4.1 *If A is a partial groupoid satisfying* (0)*, then $(G, \circ)$ is a PLRI groupoid.*

Proof. Let $a \in G$. If a^2 is not defined in A, then by (ii) we obtain that $(a \circ a) \circ a = a \circ a$, $a \circ (a \circ a) = a \circ a$, $(a \circ a) \circ (a \circ a) = a \circ a$. If a^2 is defined in A, then by (0) and (i) we obtain that $(a \circ a) \circ a = a^2 \circ a = a^2a = a^2$, $a \circ (a \circ a) = a \circ a^2 = aa^2 = a^2$, $(a \circ a) \circ (a \circ a) = a^2 \circ a^2 = a^2a^2 = a^2$. Therefore, $(G, \circ)$ is a PLRI groupoid.

Corollary 4.2 *A partial groupoid is strongly embeddable into p$\mathcal{V}$-groupoid if and only if it satisfies the condition* (0)*.*

As a special case of the Evans's Theorem ([3]) we obtain the following

Theorem 4.3 *If every partial p$\mathcal{V}$-groupoid is embeddable into a p$\mathcal{V}$-groupoid, then the word problem is solvable for the variety p$\mathcal{V}$.*

As a corollary, we have the following

Theorem 4.4 *The word problem for the variety p$\mathcal{V}$ is solvable.*

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Received: December 20, 2007