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ON THE $(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma)$-DIAPHONY OF THE NETS OF TYPE OF ZAREMBA–HALTON CONSTRUCTED IN GENERALIZED NUMBER SYSTEM

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ABSTRACT. In the present paper the so-called $(\mathrm{Vil}_{\mathcal{B}_s};\alpha;\gamma)$-diaphony as a quantitative measure for the distribution of sequences and nets is considered. A class of two-dimensional nets $Z^{\kappa,\mu}_{\mathcal{B}_2,\nu}$ of type of Zaremba-Halton constructed in a generalized $\mathcal{B}_2$-adic system or Cantor system is introduced and the $(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma)$-diaphony of these nets is studied. The influence of the vector $\alpha=(\alpha_1,\alpha_2)$ of exponential parameters to the exact order of the $(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma)$-diaphony of the nets $Z^{\kappa,\mu}_{\mathcal{B}_2,\nu}$ is shown. If $\alpha_1=\alpha_2$, then the following holds: if $1<\alpha_2<2$ the exact order is $\mathcal{O}\big(\frac{\sqrt{\log N}}{N^{1-\varepsilon}}\big)$ for some $\varepsilon>0$, if $\alpha_2=2$ the exact order is $\mathcal{O}\big(\frac{\sqrt{\log N}}{N}\big)$ and if $\alpha_2>2$ the exact order is $\mathcal{O}\big(\frac{\sqrt{\log N}}{N^{1+\varepsilon}}\big)$ for some $\varepsilon>0$. If $\alpha_1>\alpha_2$, then the following holds: if $1<\alpha_2<2$ the exact order is $\mathcal{O}\big(\frac{1}{N^{1-\varepsilon}}\big)$ for some $\varepsilon>0$, if $\alpha_2=2$ the exact order is $\mathcal{O}\big(\frac{1}{N}\big)$ and if $\alpha_2>2$ the exact order is $\mathcal{O}\big(\frac{1}{N^{1+\varepsilon}}\big)$ for some $\varepsilon>0$. Here $N=B_\nu$, where B_ν denotes the number of the points of the nets $Z^{\kappa,\mu}_{\mathcal{B}_2,\nu}$.

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1. Introduction

The functions of some complete orthonormal systems on the $s-$dimensional unit cube $[0,1)^s$ are used with a big success as an analytical tool for investigation of the distribution of sequences and nets. The first example of these systems is the trigonometric function systems. For an arbitrary integer k the function $e_k : [0,1) \to \mathbb{C}$ is defined as $e_k(x) = e^{2\pi \mathbf{i} kx}, \quad x \in [0,1)$. For an arbitrary vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{Z}^s$ the function $e_{\mathbf{k}} : [0,1)^s \to \mathbb{C}$ is defined as $e_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} e_{k_j}(x_j), \mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$. The set $\mathcal{T}_s = \{e_{\mathbf{k}}(\mathbf{x}) : \mathbf{k} \in \mathbb{Z}^s, \mathbf{x} \in [0,1)^s\}$ is called trigonometric function system.

The second example is the system of the so-called Walsh functions in base b. So, let $b \geq 2$ be a given integer. For an arbitrary integer $k \geq 0$ and a real $x \in [0,1)$ with the b-adic representations $k = \sum_{i=0}^{\nu} k_i b^i$ and $x = \sum_{i=0}^{\infty} x_i b^{-i-1}$, where $k_i, x_i \in \{0, 1, \ldots, b-1\}$, $k_\nu \neq 0$ and for infinitely many of i $x_i \neq b-1$, the kth Walsh function is defined as ${}_b\mathrm{wal}_k(x) = e^{\frac{2\pi \mathbf{i}}{b}(k_0 x_0 + \cdots + k_\nu x_\nu)}$.

Let us signify $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For an arbitrary vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ the function ${}_b\mathrm{wal}_{\mathbf{k}} : [0,1)^s \to \mathbb{C}$ is defined as ${}_b\mathrm{wal}_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} {}_b\mathrm{wal}_{k_j}(x_j)$, $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$. The set $\mathcal{W}(b) = \{{}_b\mathrm{wal}_{\mathbf{k}}(\mathbf{x}) : \mathbf{k} \in \mathbb{N}_0^s, \mathbf{x} \in [0,1)^s\}$ is called a system of Walsh functions in base b and was proposed by Chrestenson [2] and $\mathcal{W}(2)$ is the original system of the Walsh [19] functions.

Yordzhev [21–23] essentially used the b-adic arithmetic to solve some combinatorial problems.

The numerical measures for the irregularity of the distribution of sequences and nets in $[0,1)^s$ give the order of the inevitable deviation of a concrete distribution from the ideal uniform distribution. Generally, they are different kinds of the discrepancy and the diaphony. So, let $\xi_N = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1}\}$ be an arbitrary net of $N \geq 1$ points in $[0,1)^s$. For an arbitrary vector $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$ let us signify $[\mathbf{0}, \mathbf{x}) = [0, x_1) \times \cdots \times [0, x_s)$ and

$$A\big(\xi_N; [\mathbf{0}, \mathbf{x})\big) = \#\{\mathbf{x}_n : 0 \leq n \leq N-1, \mathbf{x}_n \in [\mathbf{0}, \mathbf{x})\}.$$

The star-discrepancy and the L_2-discrepancy of the net ξ_N are defined, respectively, as

$$D^*(\xi_N) = \sup_{[\mathbf{0},\mathbf{x}) \subseteq [0,1)^s} \left| \frac{A(\xi_N; [\mathbf{0}, \mathbf{x}))}{N} - \prod_{j=1}^{s} x_j \right|$$

and

$$T(\xi_N) = \left(\int_{[\mathbf{0},\mathbf{1}]^s} \left| \frac{A(\xi_N; [\mathbf{0}, \mathbf{x}))}{N} - \prod_{j=1}^{s} x_j \right|^2 \mathrm{d}x_1 \cdots \mathrm{d}x_s \right)^{\frac{1}{2}}.$$

In 1976 Zinterhof [24] introduced the notion of the so-called diaphony. So, the diaphony of the net ξ_N is defined as

$$F(\mathcal{T}_s;\xi_N)=\left(\sum_{\mathbf{k}\in\mathbb{Z}^s\setminus\{0\}} R^{-2}(\mathbf{k})\left|\frac{1}{N}\sum_{n=0}^{N-1} e_{\mathbf{k}}(\mathbf{x}_n)\right|^2\right)^{\frac{1}{2}},$$

where for each vector $\mathbf{k}=(k_1,\dots,k_s)\in\mathbb{Z}^s$ the coefficient $R(\mathbf{k})=\prod_{j=1}^s R(k_j)$ and for an arbitrary integer k

$$R(k)=\begin{cases} 1 & \text{if} \quad k=0,\\ |k| & \text{if} \quad k\neq 0.\end{cases}$$

In order to study sequences and nets constructed in b-adic system, in the last twenty years, some new versions of the diaphony was defined. These kinds of the diaphony are based on using complete orthonormal function systems constructed also in base b. In 2001 Grozdanov and Stoilova [7] used the system $\mathcal{W}(b)$ of the Walsh function to introduce the concept the so-called b-adic diaphony. So, the b-adic diaphony of the net ξ_N is defined as

$$F\big(\mathcal{W}(b);\xi_N\big)=\left(\frac{1}{(b+1)^s-1}\sum_{\mathbf{k}\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}}\rho(\mathbf{k})\left|\frac{1}{N}\sum_{n=0}^{N-1}{}_b\mathrm{wal}_{\mathbf{k}}(\mathbf{x}_n)\right|^2\right)^{\frac{1}{2}},$$

where for a vector $\mathbf{k}=(k_1,\dots,k_s)\in\mathbb{N}_0^s$ the coefficient $\rho(\mathbf{k})=\prod_{j=1}^s\rho(k_j)$ and for an arbitrary non-negative integer k

$$\rho(k)=\begin{cases} 1 & \text{if} \quad k=0,\\ b^{-2g} & \text{if} \quad b^g\le k\le b^{g+1}, g\ge 0, g\in\mathbb{Z}.\end{cases}$$

The diaphony $F(\mathcal{W}(2);\xi_N)$ which is based on using the system $\mathcal{W}(2)$ is called dyadic diaphony and was introduced by Hellekalek and Leeb [11] .

In 1954 Roth [16] obtained a general lower bound of the L_2-discrepancy of an arbitrary net. He proved that for any net ξ_N composed of N points in $[0,1)^s$ the lower bound

$$T(\xi_N)>C(s)\frac{(\log N)^{\frac{s-1}{2}}}{N} \tag{1}$$

holds, where $C(s)$ is a positive constant depending only on the dimension s.

In 1986 Proinov [15] obtained a general lower bound of the diaphony of an arbitrary net. So, for any net ξ_N composed of N points in $[0,1)^s$ the lower bound

$$F\big(\mathcal{T}_s;\xi_N\big)>\alpha(s)\frac{(\log N)^{\frac{s-1}{2}}}{N} \tag{2}$$

holds, where $\alpha(s)$ is a positive constant depending only on the dimension s.

Cristea and Pillichshammer [3] obtained a general lower bound of the b-adic diaphony of an arbitrary net. So, for any net ξ_N composed of N points in $[0,1)^s$ the lower bound

$$F\big(\mathcal{W}(b);\xi_N\big) \geq C(b,s)\frac{(\log N)^{\frac{s-1}{2}}}{N} \tag{3}$$

holds, where $C(b,s)$ is a positive constant depending on the base b and the dimension s.

Now we will present the constructions of objects for which usually the names low-discrepancy sequences and nets are used. The most famous examples was proposed by Van der Corput [17] and Halton [9]. So, let $b \geq 2$ be a given integer. For an arbitraryinteger $i \geq 0$ with the b-adic representation $i = \sum_{j=0}^{m} i_j b^j$, where $i_j \in \{0,1,\ldots,b-1\}$, $i_m \neq 0$ we put $p_b(i) = \sum_{j=0}^{m} \frac{i_j}{b^{j+1}}$. The sequence $\big(p_b(i)\big)_{i\geq 0}$ is called Van der Corput-Halton sequence.

In another direction of effort to improve the construction of the Halton sequences, several researches have studied various ways of generalizing its definition. The first technique is to include permutations, chosen either deterministically or randomly, in the radical-inverse function. This idea was developed by Faure [5,6]. Let $\Sigma = (\sigma_j)_{j\geq 0}$ be an arbitrary sequence of permutations of the set of the b-adic integers $\{0,1,\ldots,b-1\}$ which fix 0, so for $j \geq 0$ $\sigma_j(0) = 0$. The generalized radical inverse functions $p_b^{\Sigma} : \mathbb{N}_0 \to [0,1)$ with respect to the sequence Σ is defined by $p_b^{\Sigma}\big(\sum_{j=0}^{m} i_j b^j\big) = \sum_{j=0}^{m} \frac{\sigma_j(i_j)}{b^{j+1}}$. The sequence $\big(p_b^{\Sigma}(i)\big)_{i\geq 0}$ is called generalized Van der Corput sequence in base b with respect to the sequence Σ.

The sequence $\big(p_b(i)\big)_{i\geq 0}$ was used for many purposes, especially to construct two-dimensional nets. Let $b \geq 2$ and $\nu > 0$ be fixed integers. For an arbitrary integer i such that $0 \leq i \leq b^\nu - 1$ we denote $\eta_b(i) = \frac{i}{b^\nu}$. The net $R_{b,\nu} = \big\{\big(\eta_b(i), p_b(i)\big) : 0 \leq i \leq b^\nu - 1\big\}$ is called a net of Roth [16] in base b.

Roth proved that the L_2-discrepancy $T(R_{2,\nu})$ of the net $R_{2,\nu}$ have an exact order $\mathcal{O}(\frac{\log N}{N})$, where $N = 2^\nu$. According to the lower bound (1) the order $\mathcal{O}\big(\frac{\log N}{N}\big)$ is not the best possible.

There exist two important techniques to improve the order of the L_2- discrepancy of arbitrary net. They are a digital shift and a symmetrization of the points of net. The first approach was realized in 1969 by Halton and Zaremba [10]. For a given integer $\nu \geq 1$, Halton and Zaremba used the original net of Van der Corput $\{p_2(i) = 0.i_0 i_1 \ldots i_{\nu-1} : i_j \in \{0,1\},\ 0 \leq j \leq \nu-1\}$ and change the digits i_j that stay in the even positions with the digits $1 - i_j$. In this way the net $\{z_2(i) = 0.(1-i_0)i_1(1-i_2)\ldots : i_j \in \{0,1\},\ 0 \leq j \leq \nu-1\}$ is obtained.

The net $Z_{2,\nu} = \big\{\big(\eta_2(i), z_2(i)\big) : 0 \le i \le 2^\nu - 1\big\}$ which now usually is called a net of Zaremba–Halton is constructed. It is shown that the L_2–discrepancy of the net $Z_{2,\nu}$ has an exact order $\mathcal{O}\big(\frac{\sqrt{\log N}}{N}\big)$, $N = 2^\nu$, which of course is the best possible.

In 1998 Xiao [20] proved the exactness of the lower bound (2) of the diaphony for dimension $s = 2$. It is shown that the diaphony of the nets of Roth $R_{b,\nu}$ and Zaremba–Halton $Z_{b,\nu}$, both constructed in arbitrary base $b \ge 2$, have an exact order $\mathcal{O}\big(\frac{\sqrt{\log N}}{N}\big)$, $N = b^\nu$.

Grozdanov and Stoilova [7] proved the exactness of the lower bound (3) of the b-adic diaphony for dimension $s = 2$. They proved that the b-adic diaphony of the nets $R_{b,\nu}$ and $Z_{b,\nu}$ haves an exact order $\mathcal{O}\big(\frac{\sqrt{\log N}}{N}\big)$, $N = b^\nu$.

The rest of the paper is organized in the following manner: In Section 2 some preliminary notations are presented. In Section 3 the construction of the nets $Z^{\kappa,\mu}_{\mathcal{B}_2,\nu}$ of type of Zaremba–Halton in generalized $\mathcal{B}_2$-adic system is proposed and the distribution of concrete net is graphically illustrated. The main results are presented in Section 4. In Theorem 1 upper and lower bounds of the $(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma)$-diaphony of the nets $Z^{\kappa,\mu}_{\mathcal{B}_2,\nu}$ are exposed. Theorem 2 states the asymptotic behaviour of the $(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma)$-diaphony of the nets $Z^{\kappa,\mu}_{\mathcal{B}_2,\nu}$. In Section 5 the main results are proved.

2. Some preliminary notations

In 1947 Vilenkin [18] proposed new complete orthonormal function system defined in generalized number system. We will remind the construction of the functions of this system. Let the sequence of integers $B = \big\{b_0, b_1, b_2, \ldots : b_i \ge 2$ for $i \ge 0\big\}$ be given. The so-called generalized powers are defined by the next recursive manner. We put $B_0 = 1$ and for $j \ge 0$ define $B_{j+1} = B_j.b_j$.

Definition 1. For an arbitrary integer $k \ge 0$ and a real $x \in [0, 1)$ which in the so-called generalized B-adic number system or Cantor system have the representations of the form $k = \sum_{i=0}^{\nu} k_i B_i$ and and $x = \sum_{i=0}^{\infty} \frac{x_i}{B_{i+1}}$, where for $i \ge 0$, $k_i, x_i \in \{0, 1, \ldots, b_i - 1\}$, $k_\nu \ne 0$ and for infinitely many i we have $x_i \ne b_i - 1$, the kth Vilenkin function ${}_B\mathrm{Vil}_k : [0, 1) \to \mathbb{C}$ is defined as ${}_B\mathrm{Vil}_k(x) = \prod_{j=0}^{\nu} e^{2\pi \mathbf{i} \frac{x_j k_j}{b_j}}$.

We will present the multidimensional version of the Vilenkin functions. For $1 \le j \le s$ let $B_j = \big\{b_0^{(j)}, b_1^{(j)}, b_2^{(j)}, \ldots : b_i^{(j)} \ge 2$ for $i \ge 0\big\}$ be given s sequences of integer numbers. Let us denote $\mathcal{B}_s = (B_1, \ldots, B_s)$.

Definition 2. For an arbitrary vector $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$ the $\mathbf{k}$th function of Vilenkin ${}_{\mathcal{B}_s}\mathrm{Vil}_{\mathbf{k}} : [0,1)^s \to \mathbb{C}$ is defined as ${}_{\mathcal{B}_s}\mathrm{Vil}_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} {}_{B_j}\mathrm{Vil}_{k_j}(x_j)$, $\mathbf{x} = (x_1, \dots, x_s) \in [0,1)^s$. The system $\mathrm{Vil}_{\mathcal{B}_s} = \{{}_{\mathcal{B}_s}\mathrm{Vil}_{\mathbf{k}}(\mathbf{x}) : \mathbf{k} \in \mathbb{N}_0^s, \ \mathbf{x} \in [0,1)^s\}$ is called Vilenkin function system.

The function system $\mathrm{Vil}_{\mathcal{B}_s}$ was introduced by Vilenkin and independently from him this system was proposed by Price [14] in 1957. For the system $\mathrm{Vil}_{\mathcal{B}_s}$ also the name a multiplicative system is used, see Hewitt and Ross [12].

We will present one property of the functions of the system $\mathrm{Vil}_{\mathcal{B}_s}$. For arbitrary reals $x, y \in [0,1)$ with the B-adic representations $x = \sum_{i=0}^{\infty} \frac{x_i}{B_{i+1}}$ and $y = \sum_{i=0}^{\infty} \frac{y_i}{B_{i+1}}$, where for $i \geq 0$ $x_i, y_i \in \{0, 1, \dots, b_i - 1\}$ and for infinitely many i we have $x_i, y_i \neq b_i - 1$, we define the operation

$$x \oplus_B^{[0,1)} y = \left(\sum_{i=0}^{\infty} \frac{x_i + y_i \pmod{b_i}}{B_{i+1}} \right) \pmod 1.$$

For arbitrary vectors $\mathbf{x} = (x_1, \dots, x_s) \in [0,1)^s$ and $\mathbf{y} = (y_1, \dots, y_s) \in [0,1)^s$ we define the $\mathcal{B}_s$-adic operation $\mathbf{x} \oplus_{\mathcal{B}_s}^{[0,1)^s} \mathbf{y} = (x_1 \oplus_{B_1}^{[0,1)} y_1, \dots, x_s \oplus_{B_s}^{[0,1)} y_s)$. Then, for each vector $\mathbf{k} \in \mathbb{N}_0^s$ the equality holds

$$ {}_{\mathcal{B}_s}\mathrm{Vil}_{\mathbf{k}}\big(\mathbf{x} \oplus_{\mathcal{B}_s}^{[0,1)^s} \mathbf{y}\big) = {}_{\mathcal{B}_s}\mathrm{Vil}_{\mathbf{k}}(\mathbf{x}) . {}_{\mathcal{B}_s}\mathrm{Vil}_{\mathbf{k}}(\mathbf{y}). \tag{4}$$

We need to introduce some notations. For arbitrary reals $\alpha > 1$ and $\gamma > 0$ by using the sequence B of bases we define the coefficient

$$R(\alpha; \gamma; B; k) = \begin{cases} 1 & \text{if} \quad k = 0, \\ \frac{\gamma}{B_g^{\alpha}} & \text{if} \quad B_g \leq k < B_{g+1}, \ g \geq 0, \ g \in \mathbb{Z}. \end{cases}$$

Let us denote $\mu(\alpha; \gamma; B) = \sum_{k=1}^{\infty} R(\alpha; \gamma; B; k)$. Then, we have that

$$\mu(\alpha; \gamma; B) = \gamma \sum_{g=0}^{\infty} \frac{b_g - 1}{B_g^{\alpha - 1}}.$$

Now, let $\alpha = (\alpha_1, \dots, \alpha_s)$ and $\gamma = (\gamma_1, \dots, \gamma_s)$, where for $1 \leq j \leq s$ $\alpha_j > 1$ and $\gamma_1 \geq \dots \geq \gamma_s > 0$, be given vectors of exponential parameters and coordinate weights. For an arbitrary vector $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$ by using the set $\mathcal{B}_s$ we define the coefficient $R(\alpha; \gamma; \mathcal{B}_s; \mathbf{k}) = \prod_{j=1}^{s} R(\alpha_j; \gamma_j; B_j; k_j)$. Let us define the quality $C(\alpha; \gamma; \mathcal{B}_s) = \sum_{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\}} R(\alpha; \gamma; \mathcal{B}_s; \mathbf{k})$. In explicit form we have

$$C(\alpha; \gamma; \mathcal{B}_s) = \prod_{j=1}^{s} [1 + \mu(\alpha_j; \gamma_j; B_j)] - 1.$$

Baycheva and Grozdanov [1] proposed the concept of the $(\mathrm{Vil}_{\mathcal{B}_s}; \alpha; \gamma)$-diaphony.

Definition 3. The $(\mathrm{Vil}_{\mathcal{B}_s};\alpha;\gamma)$-diaphony of an arbitrary net $\xi_N = \{\mathbf{x}_0, \mathbf{x}_1, \ldots \ldots, \mathbf{x}_{N-1}\}$ of $N \geq 1$ points in $[0,1)^s$ is defined as

$$F(\mathrm{Vil}_{\mathcal{B}_s};\alpha;\gamma;\xi_N) = \left(\frac{1}{C(\alpha;\gamma;\mathcal{B}_s)} \sum_{\mathbf{k}\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}} R(\alpha;\gamma;\mathcal{B}_s;\mathbf{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_{\mathcal{B}_s}\mathrm{Vil}_{\mathbf{k}}(\mathbf{x}_n) \right|^2 \right)^{\frac{1}{2}},$$

where the coefficients $R(\alpha;\gamma;\mathcal{B}_s;\mathbf{k})$ and the constant $C(\alpha;\gamma;\mathcal{B}_s)$ are defined as above.

In general, the motives for us to introduce the coefficients $R(\alpha;\gamma;\mathcal{B}_s;\mathbf{k})$ are relevant to quasi-Monte Carlo integration in a function class which is a reproducing kernel Hilbert space. In fact, the $(\mathrm{Vil}_{\mathcal{B}_s};\alpha;\gamma)$-diaphony is the worst-case error of the integration. Here we will use the $(\mathrm{Vil}_{\mathcal{B}_s};\alpha;\gamma)$-diaphony as a quantitative measure for the quality of the distribution of nets.

3. The nets of type of Zaremba-Halton constructed in generalized number system

We will remind the constructive principle of the sequence of Van der Corput defined in a generalized B-adic system.

Definition 4. For an arbitrary integer $i \geq 0$ with the B-adic representation $i = i_m B_m + i_{m-1} B_{m-1} + \cdots + i_0 B_0$, where for $0 \leq j \leq m$ $\ i_j \in \{0, 1, \ldots, b_j - 1\}$ and $i_m \neq 0$, we put

$$p_B(i) = \frac{i_0}{B_1} + \frac{i_1}{B_2} + \cdots + \frac{i_m}{B_{m+1}}.$$

The sequence $\omega_B = (p_B(i))_{i\geq 0}$ is called a sequence of Van der Corput constructed in the generalized B-adic system.

The star-discrepancy of the sequence ω_B was studied by Haddley, Lertchoosakul and Nair [8] and also by Lertchoosakul and Nair [13].

We will introduce the constructive principle of two-dimensional nets of type of Zaremba-Halton. For this purpose let

$$B_1 = \left\{ b_0^{(1)}, b_1^{(1)}, \ldots, b_{\nu-1}^{(1)}, b_\nu^{(1)}, b_{\nu+1}^{(1)}, \ldots \ :\ b_j^{(1)} \geq 2 \quad \text{for} \ \ j \geq 0 \right\}$$

be a given sequence of bases. We define the sequence

$$B_2 = \left\{ b_0^{(2)}, b_1^{(2)}, \ldots, b_{\nu-1}^{(2)}, b_\nu^{(2)}, b_{\nu+1}^{(2)}, \ldots \ :\ b_j^{(2)} \geq 2 \quad \text{for} \ \ j \geq 0 \right\}$$

in the following manner. We put $b_0^{(2)} = b_{\nu-1}^{(1)}$, $b_1^{(2)} = b_{\nu-2}^{(1)}, \ldots, b_{\nu-1}^{(2)} = b_0^{(1)}$ and the bases $b_\nu^{(2)}, b_{\nu+1}^{(2)}, \ldots$ can be chosen arbitrary.

Let $\nu \geq 1$ be an arbitrary and fixed integer. We have that

$$B_\nu^{(1)} = b_0^{(1)} \cdots b_{\nu-1}^{(1)} = b_{\nu-1}^{(2)} \cdots b_0^{(1)} = B_\nu^{(2)}$$

and let us denote

$$B_\nu = B_\nu^{(1)} = B_\nu^{(2)}.$$

An arbitrary integer i such that $0 \leq i \leq B_\nu - 1$ we present in the B_2-adic system as

$$i = i_{\nu-1} B_{\nu-1}^{(2)} + i_{\nu-2} B_{\nu-2}^{(2)} + \cdots + i_1 B_1^{(2)} + i_0 B_0^{(2)},$$

where for

$$0 \leq j \leq \nu - 1, \quad i_j \in \left\{0, 1, \ldots, b_j^{(2)} - 1\right\}.$$

Then, the quantities $\eta_{B_1}(i) = \frac{i}{B_\nu}$ and $p_{B_2}(i)$ have, respectively the B_1-adic and the B_2-adic representations of the form

$$\eta_{B_1}(i) = 0.i_{\nu-1} i_{\nu-2} \ldots i_0 \quad \text{and} \quad p_{B_2}(i) = 0.i_0 i_1 \ldots i_{\nu-1}.$$

Let $\kappa = 0.\kappa_{\nu-1}\kappa_{\nu-2}\ldots\kappa_0$ and $\mu = 0.\mu_0\mu_1\ldots\mu_{\nu-1}$, where for $0 \leq j \leq \nu - 1$ $\kappa_{\nu-1-j} \in \left\{0, 1, \ldots, b_j^{(1)} - 1\right\}$ and $\mu_j \in \left\{0, 1, \ldots, b_j^{(2)} - 1\right\}$, be fixed B_1-adic and B_2-adic rational numbers. Let us define the quantities $\eta_{B_1}^{\kappa}(i) = \eta_{B_1}(i) \oplus_{B_1}^{[0,1)} \kappa$ and $z_{B_2}^{\mu}(i) = p_{B_2}(i) \oplus_{B_2}^{[0,1)} \mu$.

Definition 5. Let $\nu \geq 1$ be an arbitrary and fixed integer. Let κ and μ be as already defined. The two-dimensional net

$$Z_{\mathcal{B}_2,\nu}^{\kappa,\mu} = \left\{\left(\eta_{B_1}^{\kappa}(i), z_{B_2}^{\mu}(i)\right) : 0 \leq i \leq B_\nu - 1\right\}$$

we will call a net of type of Zaremba-Halton constructed in the generalized $\mathcal{B}_2$-adic system.

Dimitrievska Ristovska, Grozdanov and Petrova [4] presented a code of program that can construct arbitrary net of type of Zaremba-Halton. Many examples are considered and graphically illustrated. Here we will give only one example. We choose the parameters in the following manner:

$$\nu = 6,$$

$$b_0^{(1)} = 2, \quad b_1^{(1)} = 3, \quad b_2^{(1)} = 4, \quad b_3^{(1)} = 5, \quad b_4^{(1)} = 6, \quad b_5^{(1)} = 8,$$

$$b_0^{(2)} = 8, \quad b_1^{(2)} = 6, \quad b_2^{(2)} = 5, \quad b_3^{(2)} = 4, \quad b_4^{(2)} = 3, \quad b_5^{(2)} = 2,$$

$$\kappa = 0.121071 \quad \text{and} \quad \mu = 0.750121.$$

The distribution of the points of the net $Z_{\mathcal{B}_2,\nu}^{\kappa,\mu}$ is given in Figure 1.

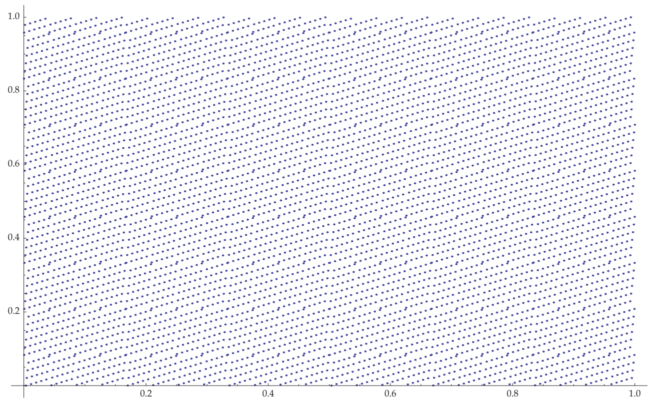

Figure 1.

4. Statements of the main results

Here we will present the main results of the paper. They are lower and upper bounds of the $(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma)$-diaphony of the nets $Z^{\kappa,\mu}_{\mathcal{B}_2,\nu}$ of type of Zaremba– Halton. These bounds will permit us to obtain the exact orders and the asymptotic behavior of the $(Vil_{\mathcal{B}_2};\alpha;\gamma)$-diaphony of the nets $Z^{\kappa,\mu}_{\mathcal{B}_2,\nu}$. The influence of the vector α of exponential parameters to the exact orders of the $(Vil_{\mathcal{B}_2};\alpha;\gamma)$-diaphony of the nets $Z^{\kappa,\mu}_{\mathcal{B}_2,\nu}$ will be shown.

Theorem 1. *Let us suppose that the sequences B_1 and B_2 of bases are limited from above, i.e., there exists a constant $M \geq 2$ such that for each $j \geq 0$ and $\tau = 1, 2$ we have $b^{(\tau)}_j \leq M$.*

Let $Z^{\kappa,\mu}_{\mathcal{B}_2,\nu}$ be an arbitrary net of type of Zaremba–Halton. Then, there exist constants C'_τ and C_τ, $\tau = 1, 2, 3, 4$, such that the $(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma)$-diaphony of the net $Z^{\kappa,\mu}_{\mathcal{B}_2,\nu}$ satisfies the inequalities

$$
\begin{aligned}
&C'_1 \frac{1}{B_\nu^{\alpha_2}} \sum_{g=0}^{\nu-1} \big[B_g^{(1)}\big]^{\alpha_2-\alpha_1} + C'_2 \frac{1}{B_\nu^{\alpha_1}} + C'_3 \frac{1}{B_\nu^{\alpha_2}} + C'_4 \frac{1}{B_\nu^{\alpha_1+\alpha_2-1}} \\
&\leq F^2(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z^{\kappa,\mu}_{\mathcal{B}_2,\nu}) < C_1 \frac{1}{B_\nu^{\alpha_2}} \sum_{g=0}^{\nu-1} \big[B_g^{(1)}\big]^{\alpha_2-\alpha_1} \\
&\quad + C_2 \frac{1}{B_\nu^{\alpha_1}} + C_3 \frac{1}{B_\nu^{\alpha_2}} + C_4 \frac{1}{B_\nu^{\alpha_1+\alpha_2-1}}.
\end{aligned}
$$

The constants C'_τ and C_τ will be obtained in explicit form within the proof of the theorem.

In the next theorem we will present the exact orders of the $(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma)$--diaphony of the nets $Z^{\kappa,\mu}_{\mathcal{B}_2,\nu}$ of the type of Zaremba–Halton. These exact orders depend on the exponential parameters α_1 and α_2. So, the following theorem holds.

Theorem 2. *Let the conditions of Theorem 1 be realized. Let $Z^{\kappa,\mu}_{\mathcal{B}_2,\nu}$ be an arbitrary net of type of Zaremba–Halton. Let us denote $B_\nu = N$. Then, the diaphony $F(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z^{\kappa,\mu}_{\mathcal{B}_2,\nu}) = F(\cdot)$ satisfies the following:*

(i) *If $\alpha_1 = \alpha_2$, then $F(\cdot) = \mathcal{O}\Big(\frac{\sqrt{\log N}}{N^{\frac{\alpha_2}{2}}}\Big)$;*

(i1) *If $1 < \alpha_2 < 2$, then there exists some $\varepsilon > 0$ such $F(\cdot) = \mathcal{O}\big(\frac{\sqrt{\log N}}{N^{1-\varepsilon}}\big)$;*

(i2) *If $\alpha_2 = 2$, then $F(\cdot) = \mathcal{O}\big(\frac{\sqrt{\log N}}{N}\big)$;*

(i3) *If $\alpha_2 > 2$, then there exists some $\varepsilon > 0$ such that $F(\cdot) = \mathcal{O}\big(\frac{\sqrt{\log N}}{N^{1+\varepsilon}}\big)$;*

(ii) *If $\alpha_1 > \alpha_2$, then $F(\cdot) = \mathcal{O}\Big(\frac{1}{N^{\frac{\alpha_2}{2}}}\Big)$;*

(ii1) *If $1 < \alpha_2 < 2$, then there exists some $\varepsilon > 0$ such that $F(\cdot) = \mathcal{O}\big(\frac{1}{N^{1-\varepsilon}}\big)$;*

(ii2) *If $\alpha_2 = 2$, then $F(\cdot) = \mathcal{O}\big(\frac{1}{N}\big)$;*

(ii3) *If $\alpha_2 > 2$, then there exists some $\varepsilon > 0$ such that $F(\cdot) = \mathcal{O}\big(\frac{1}{N^{1+\varepsilon}}\big)$.*

5. Proofs of the main results

Some preliminary results: To prove Theorems 1 and 2 we need to obtain the exact values of the trigonometric sums of the net $\{\eta_{B_1}(i) : 0 \le i \le B_\nu - 1\}$ with respect to the system Vil_{B_1}, of the net $\{p_{B_2}(i) : 0 \le i \le B_\nu - 1\}$ with respect to the system Vil_{B_2} and of the two-dimensional net $\big\{\big(\eta_{B_1}(i), p_{B_2}(i)\big) : 0 \le i \le B_\nu - 1\big\}$ with respect to the system $\mathrm{Vil}_{\mathcal{B}_2}$.

Lemma 1. *Let $\nu \ge 1$ be an arbitrary fixed integer. Then, the following holds:*

(i) *For each integers g, k and l such that $0 \le g \le \nu-1$, $B^{(1)}_g \le k \le B^{(1)}_{g+1} - 1$ and $B^{(2)}_g \le l \le B^{(2)}_{g+1} - 1$ the equalities hold*

$$\sum_{i=0}^{B_\nu - 1} {}_{B_1}\mathrm{Vil}_k\big(\eta_{B_1}(i)\big) = 0 \quad \text{and} \quad \sum_{i=0}^{B_\nu - 1} {}_{B_2}\mathrm{Vil}_l\big(p_{B_2}(i)\big) = 0.$$

(ii) *For each integers g, k and l such that $g \geq \nu$, $B_g^{(1)} \leq k \leq B_{g+1}^{(1)} - 1$ and $B_g^{(2)} \leq l \leq B_{g+1}^{(2)} - 1$ the equalities hold*

$$\left|\sum_{i=0}^{B_\nu-1} {}_{B_1}\mathrm{Vil}_k\big(\eta_{B_1}(i)\big)\right| = \left|\sum_{i=0}^{B_\nu-1} {}_{B_2}\mathrm{Vil}_l\big(p_{B_2}(i)\big)\right| = \begin{cases} B_\nu & \text{if } k(l) \equiv 0 \pmod{B_\nu}, \\ 0 & \text{if } k(l) \not\equiv 0 \pmod{B_\nu}. \end{cases}$$

Proof.

(i) Let $0 \leq g \leq \nu-1$ be a fixed. Let an arbitrary integer $B_g^{(1)} \leq k \leq B_{g+1}^{(1)}-1$ have the B_1-adic representation $k = k_0B_0^{(1)}+k_1B_1^{(1)}+\cdots+k_gB_g^{(1)}$, where for $0 \leq j \leq g$, $k_j \in \{0, 1, \ldots, b_j^{(1)} - 1\}$ and $k_g \neq 0$. Then, we have

$$\sum_{i=0}^{B_\nu-1} {}_{B_1}\mathrm{Vil}_k\big(\eta_{B_1}(i)\big) = \sum_{i_{\nu-1}=0}^{b_0^{(1)}-1} e^{2\pi\mathbf{i}\frac{k_0 i_{\nu-1}}{b_0^{(1)}}} \cdots \sum_{i_{\nu-1-g}=0}^{b_g^{(1)}-1} e^{2\pi\mathbf{i}\frac{k_g i_{\nu-1-g}}{b_g^{(1)}}} \sum_{i_{\nu-2-g}=0}^{b_{g+1}^{(1)}-1} 1 \cdots \sum_{i_0=0}^{b_{\nu-1}^{(1)}-1} 1 = 0.$$

(ii) Let $g \geq \nu$ be a fixed and $B_g^{(1)} \leq k \leq B_{g+1}^{(1)} - 1$ be an arbitrary integer. Let us assume that $k \equiv 0(\mathrm{mod}\ B_\nu)$. Then, k has the B_1-adic representation

$$k = k_gB_g^{(1)} + \cdots + k_\nu B_\nu^{(1)},$$

hence we have that

$${}_{B_1}\mathrm{Vil}_k\big(\eta_{B_1}(i)\big) = \prod_{j=0}^{\nu-1} e^{2\pi\mathbf{i}\frac{0.i_{\nu-1-j}}{b_j^{(1)}}} = 1 \quad \text{and} \quad \sum_{i=0}^{B_\nu-1} {}_{B_1}\mathrm{Vil}_k\big(\eta_{B_1}(i)\big) = B_\nu.$$

Let us assume that $k \not\equiv 0(\mathrm{mod}\ B_\nu)$. Then, k has the B_1-adic representation of the form $k = k_gB_g^{(1)} + \cdots + k_\nu B_\nu^{(1)} + k_{\nu-1}B_{\nu-1}^{(1)} + \cdots + k_1B_1^{(1)} + k_0B_0^{(1)}$, where there exists at least one index δ, $0 \leq \delta \leq \nu - 1$ such that $k_\delta \neq 0$. Hence, we obtain that

$$\sum_{i=0}^{B_\nu-1} {}_{B_1}\mathrm{Vil}_k\big(\eta_{B_1}(i)\big) = \sum_{i_{\nu-1}=0}^{b_0^{(1)}-1} e^{2\pi\mathbf{i}\frac{k_0 i_{\nu-1}}{b_0^{(1)}}} \cdots \sum_{i_{\nu-1-\delta}=0}^{b_\delta^{(1)}-1} e^{2\pi\mathbf{i}\frac{k_\delta i_{\nu-1-\delta}}{b_\delta^{(1)}}} \cdots \sum_{i_0=0}^{b_{\nu-1}^{(1)}-1} e^{2\pi\mathbf{i}\frac{k_{\nu-1} i_0}{b_{\nu-1}^{(1)}}} = 0. \quad \square$$

The other equalities of the lemma can be proved by similar manner.

Lemma 2. *Let $\nu \geq 1$ be an arbitrary fixed integer. Then, the following holds:*
(i) *Let the integers g_1 and g_2 such that $0 \leq g_1 \leq \nu - 1$ and $0 \leq g_2 \leq \nu - 1$ be arbitrary. For an arbitrary integer $0 \leq g \leq g_1$ let us define the set*

$$A(g_1; g) = \{\ k_1 : k_1 = k_g^{(1)} B_g^{(1)} + k_{g+1}^{(1)} B_{g+1}^{(1)} + \cdots + k_{g_1}^{(1)} B_{g_1}^{(1)},$$

for

$$g \leq j \leq g_1, \quad k_j^{(1)} \in \{0, 1, \ldots, b_j^{(1)} - 1\}, \quad k_g^{(1)} \neq 0, \quad k_{g_1}^{(1)} \neq 0\ \}.$$

For an arbitrary integer $k_1 \in A(g_1; g)$ let us define the B_2-adic integer

$$\overline{k}_1 = \overline{k}_{g_1}^{(1)} B_{\nu-1-g_1}^{(2)} + \overline{k}_{g_1-1}^{(1)} B_{\nu-g_1}^{(2)} + \cdots + \overline{k}_g^{(1)} B_{\nu-1-g}^{(2)},$$

where for $g \leq j \leq g_1$ we put $\overline{k}_j^{(1)} = b_j^{(1)} - k_j^{(1)} \pmod{b_j^{(1)}}$. Then, for each integer k_2 such that $B_{g_2}^{(2)} \leq k_2 \leq B_{g_2+1}^{(2)} - 1$ the equalities hold

$$\left| \sum_{i=0}^{B_\nu - 1} {}_{B_1}\mathrm{Vil}_{k_1}\big(\eta_{B_1}(i)\big) \cdot {}_{B_2}\mathrm{Vil}_{k_2}\big(p_{B_2}(i)\big) \right| = \begin{cases} B_\nu & \text{if} \quad k_2 = \overline{k}_1, \\ 0 & \text{if} \quad k_2 \neq \overline{k}_1. \end{cases}$$

In the case when $k_2 = \overline{k}_1$ we have that $g_2 = \nu - 1 - g$.
(ii) *Let the integers g_1 and g_2 such that $0 \leq g_1 \leq \nu - 1$ and $g_2 \geq \nu$ be arbitrary. For an arbitrary integer k_1, $B_{g_1}^{(1)} \leq k_1 \leq B_{g_1+1}^{(1)} - 1$ we will use the B_1-adic representation $k_1 = \sum_{j=0}^{\nu-1} k_j^{(1)} B_j^{(1)}$, where for $0 \leq j \leq \nu - 1$ $k_j^{(1)} \in \{0, 1, \ldots, b_j^{(1)} - 1\}$. By using the integer k_1 let us define the B_2-adic integer $\overline{k}_1 = \sum_{j=0}^{\nu-1} \overline{k}_{\nu-1-j}^{(1)} B_j^{(2)}$, where for $0 \leq j \leq \nu - 1$ we put $\overline{k}_j^{(1)} = b_j^{(1)} - k_j^{(1)} \pmod{b_j^{(1)}}$.*

An arbitrary integer k_2, $B_{g_2}^{(2)} \leq k_2 \leq B_{g_2+1}^{(2)} - 1$ we will present in the B_2-adic form $k_2 = \sum_{j=0}^{\nu-1} k_j^{(2)} B_j^{(2)} + \sum_{j=\nu}^{g_2} k_j^{(2)} B_j^{(2)} = k_2' + k_2''$. Then, the equalities hold

$$\left| \sum_{i=0}^{B_\nu - 1} {}_{B_1}\mathrm{Vil}_{k_1}\big(\eta_{B_1}(i)\big) \cdot {}_{B_2}Vil_{k_2}\big(p_{B_2}(i)\big) \right| = \begin{cases} B_\nu & \text{if} \quad k_2' = \overline{k}_1, \\ 0 & \text{if} \quad k_2' \neq \overline{k}_1. \end{cases}$$

(iii) *Let the integers g_1 and g_2 such that $0 \leq g_2 \leq \nu - 1$ and $g_1 \geq \nu$ be arbitrary. For an arbitrary integer k_2, $B_{g_2}^{(2)} \leq k_2 \leq B_{g_2+1}^{(2)} - 1$ we will use the B_2-adic representation $k_2 = \sum_{j=0}^{\nu-1} k_j^{(2)} B_j^{(2)}$, where for $0 \leq j \leq \nu - 1$ $k_j^{(2)} \in \{0, 1, \ldots, b_j^{(2)} - 1\}$. By using the integer k_2 let us define the B_1-adic integer $\overline{k}_2 = \sum_{j=0}^{\nu-1} \overline{k}_{\nu-1-j}^{(2)} B_j^{(1)}$, where for $0 \leq j \leq \nu - 1$ $\overline{k}_j^{(2)} = b_j^{(2)} - k_j^{(2)} \pmod{b_j^{(2)}}$.*

An arbitrary integer k_1, $B^{(1)}_{g_1} \le k_1 \le B^{(1)}_{g_1+1} - 1$ *we will present in the* B_1*-adic form* $k_1 = \sum_{j=0}^{\nu-1} k^{(1)}_j B^{(1)}_j + \sum_{j=\nu}^{g_1} k^{(1)}_j B^{(1)}_j = k'_1 + k''_1$. *Then, the equalities hold*

$$\left|\sum_{i=0}^{B_\nu-1} {}_{B_1}\mathrm{Vil}_{k_1}\big(\eta_{B_1}(i)\big) \cdot {}_{B_2}Vil_{k_2}\big(p_{B_2}(i)\big)\right| = \begin{cases} B_\nu & \text{if} \quad k'_1 = \overline{k}_2, \\ 0 & \text{if} \quad k'_1 \ne \overline{k}_2. \end{cases}$$

(iv) *Let the integers* g_1 *and* g_2 *such that* $g_1 \ge \nu$ *and* $g_2 \ge \nu$ *be arbitrary. An arbitrary integer* k_1, $B^{(1)}_{g_1} \le k_1 \le B^{(1)}_{g_1+1} - 1$ *we will present in the* B_1*-adic form* $k_1 = \sum_{j=0}^{\nu-1} k^{(1)}_j B^{(1)}_j + \sum_{j=\nu}^{g_1} k^{(1)}_j B^{(1)}_j = k'_1 + k''_1$. *By using the integer* k'_1 *let us define the* B_2*-adic integer* $\overline{k}'_1 = \sum_{j=0}^{\nu-1} \overline{k}^{(1)}_{\nu-1-j} B^{(2)}_j$, *where for* $0 \le j \le \nu-1$ *we put* $\overline{k}^{(1)}_j = b^{(1)}_j - k^{(1)}_j \pmod{b^{(1)}_j}$.

An arbitrary integer k_2, $B^{(2)}_{g_2} \le k_2 \le B^{(2)}_{g_2+1} - 1$ *we will present in the* B_2*-adic form* $k_2 = \sum_{j=0}^{\nu-1} k^{(2)}_j B^{(2)}_j + \sum_{j=\nu}^{g_2} k^{(2)}_j B^{(2)}_j = k'_2 + k''_2$. *Then, the equalities hold*

$$\left|\sum_{i=0}^{B_\nu-1} {}_{B_1}\mathrm{Vil}_{k_1}\big(\eta_{B_1}(i)\big) \cdot {}_{B_2}Vil_{k_2}\big(p_{B_2}(i)\big)\right| = \begin{cases} B_\nu & \text{if} \quad k'_2 = \overline{k}'_1, \\ 0 & \text{if} \quad k'_2 \ne \overline{k}'_1. \end{cases}$$

P r o o f. (i) Let $k_1 \in A(g_1; g)$ be an arbitrary integer. Then, we have that

$$ {}_{B_1}\mathrm{Vil}_{k_1}\big(\eta_{B_1}(i)\big) = e^{2\pi \mathbf{i} \frac{k^{(1)}_g i_{\nu-1-g}}{b^{(1)}_g}} \cdot e^{2\pi \mathbf{i} \frac{k^{(1)}_{g+1} i_{\nu-2-g}}{b^{(1)}_{g+1}}} \cdots \\ \cdots e^{2\pi \mathbf{i} \frac{k^{(1)}_{g_1-1} i_{\nu-g_1}}{b^{(1)}_{g_1-1}}} \cdot e^{2\pi \mathbf{i} \frac{k^{(1)}_{g_1} i_{\nu-1-g_1}}{b^{(1)}_{g_1}}}. \quad (5)$$

Let us assume that the conditions $g_2 = \nu-1-g$ and $k_2 = \overline{k}_1$ hold. This means that k_2 has the B_2-adic representation

$$k_2 = k^{(2)}_{\nu-1-g_1} B^{(2)}_{\nu-1-g_1} + k^{(2)}_{\nu-g_1} B^{(2)}_{\nu-g_1} + \cdots + k^{(2)}_{\nu-1-g} B^{(2)}_{\nu-1-g},$$

where for $\nu-1-g_1 \le j \le \nu-1-g$ $k^{(2)}_j = \overline{k}^{(1)}_{\nu-1-j}$. Then, we have

$$ {}_{B_2}\mathrm{Vil}_{k_2}\big(p_{B_2}(i)\big) = \\ e^{2\pi \mathbf{i} \frac{k^{(2)}_{\nu-1-g_1} i_{\nu-1-g_1}}{b^{(2)}_{\nu-1-g_1}}} \cdot e^{2\pi \mathbf{i} \frac{k^{(2)}_{\nu-g_1} i_{\nu-g_1}}{b^{(2)}_{\nu-g_1}}} \cdots e^{2\pi \mathbf{i} \frac{k^{(2)}_{\nu-2-g} i_{\nu-2-g}}{b^{(2)}_{\nu-2-g}}} \cdot e^{2\pi \mathbf{i} \frac{k^{(2)}_{\nu-1-g} i_{\nu-1-g}}{b^{(2)}_{\nu-1-g}}} = \\ e^{2\pi \mathbf{i} \frac{\overline{k}^{(1)}_g i_{\nu-1-g}}{b^{(1)}_g}} \cdot e^{2\pi \mathbf{i} \frac{\overline{k}^{(1)}_{g+1} i_{\nu-2-g}}{b^{(1)}_{g+1}}} \cdots e^{2\pi \mathbf{i} \frac{\overline{k}^{(1)}_{g_1-1} i_{\nu-g_1}}{b^{(1)}_{g_1-1}}} \cdot e^{2\pi \mathbf{i} \frac{\overline{k}^{(1)}_{g_1} i_{\nu-1-g_1}}{b^{(1)}_{g_1}}}. \quad (6)$$

From (5) and (6) we obtain that

$$
\begin{aligned}
{}_{B_1}\operatorname{Vil}_{k_1}\big(\eta_{B_1}(i)\big)\cdot{}_{B_2}\operatorname{Vil}_{k_2}\big(p_{B_2}(i)\big) &= \\
& e^{2\pi\mathbf{i}\frac{(k_g^{(1)}+\overline{k}_g^{(1)})i_{\nu-1-g}}{b_g^{(1)}}}\cdot e^{2\pi\mathbf{i}\frac{(k_{g+1}^{(1)}+\overline{k}_{g+1}^{(1)})i_{\nu-2-g}}{b_{g+1}^{(1)}}}\cdots \\
& \cdots e^{2\pi\mathbf{i}\frac{(k_{g_1-1}^{(1)}+\overline{k}_{g_1-1}^{(1)})i_{\nu-g_1}}{b_{g_1-1}^{(1)}}}\cdot e^{2\pi\mathbf{i}\frac{(k_{g_1}^{(1)}+\overline{k}_{g_1}^{(1)})i_{\nu-1-g_1}}{b_{g_1}^{(1)}}}=1
\end{aligned}
$$

and hence $\sum_{i=0}^{B_\nu-1}{}_{B_1}\operatorname{Vil}_{k_1}\big(\eta_{B_1}(i)\big)\cdot{}_{B_2}\operatorname{Vil}_{k_2}\big(p_{B_2}(i)\big)=B_\nu$.

Let us assume that $k_2\neq\overline{k}_1$. We will consider two cases. First, let $g_2=\nu-1-g$ and there is at least one index δ, $\nu-1-g_1\leq\delta\leq\nu-1-g$ such that $k_{\nu-1-g_1}^{(2)}=\overline{k}_{g_1}^{(1)},\dots,k_{\nu-1-\delta}^{(2)}\neq\overline{k}_\delta^{(1)},\dots,k_{\nu-1-g}^{(2)}=\overline{k}_g^{(1)}$. From (5) and (6) we obtain that

$$
\begin{aligned}
\sum_{i=0}^{B_\nu-1}{}_{B_1}\operatorname{Vil}_{k_1}\big(\eta_{B_1}(i)\big)\cdot{}_{B_2}\operatorname{Vil}_{k_2}\big(p_{B_2}(i)\big) &= \sum_{i_{\nu-1}=0}^{b_0^{(1)}-1}\cdots \\
\cdots\sum_{i_{\nu-g_1}=0}^{b_{g_1-1}^{(1)}-1}\sum_{i_{\nu-1-g_1}=0}^{b_{g_1}^{(1)}-1} & e^{2\pi\mathbf{i}\frac{(k_{g_1}^{(1)}+\overline{k}_{g_1}^{(1)})i_{\nu-1-g_1}}{b_{g_1}^{(1)}}}\cdots \\
\cdots\sum_{i_{\nu-1-\delta}=0}^{b_\delta^{(1)}-1} & e^{2\pi\mathbf{i}\frac{(k_\delta^{(1)}+k_{\nu-1-\delta}^{(2)})i_{\nu-1-\delta}}{b_\delta^{(1)}}}\cdots \\
\cdots\sum_{i_{\nu-1-g}=0}^{b_g^{(1)}-1} & e^{2\pi\mathbf{i}\frac{(k_g^{(1)}+\overline{k}_g^{(1)})i_{\nu-1-g}}{b_g^{(1)}}}\sum_{i_{\nu-2-g}=0}^{b_{g+1}^{(1)}-1}1\cdots\sum_{i_0=0}^{b_{\nu-1}^{(1)}-1}1=0.
\end{aligned}
$$

Second, let us assume that $g_2\neq\nu-1-g$. We have that

$$
{}_{B_1}\operatorname{Vil}_{k_1}\big(\eta_{B_1}(i)\big)=\prod_{j=g}^{g_1}e^{2\pi\mathbf{i}\frac{k_j^{(1)}i_{\nu-1-j}}{b_{\nu-1-j}^{(2)}}}=\prod_{j=\nu-1-g_1}^{\nu-1-g}e^{2\pi\mathbf{i}\frac{k_{\nu-1-j}^{(1)}i_j}{b_j^{(2)}}}.
$$

Let an arbitrary integer k_2 such that $B_{g_2}^{(2)}\leq k_2\leq B_{g_2+1}^{(2)}-1$ have the B_2-adic representation $k_2=k_0^{(2)}B_0^{(2)}+\cdots+k_{g_2}^{(2)}B_{g_2}^{(2)}$, where $k_{g_2}^{(2)}\neq 0$. Hence, we have that

$$
{}_{B_2}\operatorname{Vil}_{k_2}\big(p_{B_2}(i)\big)=\prod_{j=0}^{g_2}e^{2\pi\mathbf{i}\frac{k_j^{(2)}i_j}{b_j^{(2)}}}.
$$

Let us assume that $0 \le g_2 < \nu - 1 - g_1$. In this case we have that

$$\sum_{i=0}^{B_\nu-1} {}_{B_1}\mathrm{Vil}_{k_1}\big(\eta_{B_1}(i)\big) \cdot {}_{B_2}\mathrm{Vil}_{k_2}\big(p_{B_2}(i)\big)$$

$$= \prod_{j=0}^{g_2} \sum_{i_j=0}^{b_j^{(2)}-1} e^{2\pi \mathbf{i} \frac{k_j^{(2)} i_j}{b_j^{(2)}}} \cdot \prod_{j=g_2+1}^{\nu-2-g_1} \sum_{i_j=0}^{b_j^{(2)}-1} 1$$

$$\times \prod_{j=\nu-1-g_1}^{\nu-1-g} \sum_{i_j=0}^{b_j^{(2)}-1} e^{2\pi \mathbf{i} \frac{k_{\nu-1-j}^{(1)} i_j}{b_j^{(2)}}} \cdot \prod_{j=\nu-g}^{\nu-1} \sum_{i_j=0}^{b_j^{(2)}-1} 1 = 0.$$

Let us assume that $\nu - 1 - g_1 \le g_2 < \nu - 1 - g$. In this case we have that

$$\sum_{i=0}^{B_\nu-1} {}_{B_1}\mathrm{Vil}_{k_1}\big(\eta_{B_1}(i)\big) \cdot {}_{B_2}\mathrm{Vil}_{k_2}\big(p_{B_2}(i)\big)$$

$$= \prod_{j=0}^{\nu-2-g_1} \sum_{i_j=0}^{b_j^{(2)}-1} e^{2\pi \mathbf{i} \frac{k_j^{(2)} i_j}{b_j^{(2)}}} \cdot \prod_{j=\nu-1-g_1}^{g_2} \sum_{i_j=0}^{b_j^{(2)}-1} e^{2\pi \mathbf{i} \frac{(k_{\nu-1-j}^{(1)} + k_j^{(2)}) i_j}{b_j^{(2)}}}$$

$$\times \prod_{j=g_2+1}^{\nu-1-g} \sum_{i_j=0}^{b_j^{(2)}-1} e^{2\pi \mathbf{i} \frac{k_{\nu-1-j}^{(1)} i_j}{b_j^{(2)}}} \cdot \prod_{j=\nu-g}^{\nu-1} \sum_{i_j=0}^{b_j^{(2)}-1} 1 = 0.$$

Let us assume that $\nu - 1 \ge g_2 > \nu - 1 - g$. In this case we have that

$$\sum_{i=0}^{B_\nu-1} {}_{B_1}\mathrm{Vil}_{k_1}\big(\eta_{B_1}(i)\big) \cdot {}_{B_2}\mathrm{Vil}_{k_2}\big(p_{B_2}(i)\big)$$

$$= \prod_{j=0}^{\nu-2-g_1} \sum_{i_j=0}^{b_j^{(2)}-1} e^{2\pi \mathbf{i} \frac{k_j^{(2)} i_j}{b_j^{(2)}}} \cdot \prod_{j=\nu-1-g_1}^{\nu-1-g} \sum_{i_j=0}^{b_j^{(2)}-1} e^{2\pi \mathbf{i} \frac{(k_{\nu-1-j}^{(1)} + k_j^{(2)}) i_j}{b_j^{(2)}}}$$

$$\times \prod_{j=\nu-g}^{g_2} \sum_{i_j=0}^{b_j^{(2)}-1} e^{2\pi \mathbf{i} \frac{k_j^{(2)} i_j}{b_j^{(2)}}} \cdot \prod_{j=g_2+1}^{\nu-1} \sum_{i_j=0}^{b_j^{(2)}-1} 1 = 0.$$

The statements (ii), (iii) and (iv) of the Lemma 2 can be proved by similar technique. Here, we will leave out the details. □

A proof of Theorem 1. We will apply the equality (4) to the trigonometric sums of the nets $\{\eta_{B_1}^{\kappa}(i) : 0 \le i \le B_\nu - 1\}$, $\{z_{B_2}^{\mu}(i) : 0 \le i \le B_\nu - 1\}$ and $Z_{\mathcal{B}_2,\nu}^{\kappa,\mu} = \big\{\big(\eta_{B_1}^{\kappa}(i), z_{B_2}^{\mu}(i)\big) : 0 \le i \le B_\nu - 1\big\}$ with respect to the functions of the systems Vil_{B_1}, Vil_{B_2} and $Vil_{\mathcal{B}_2}$. According to Definition 3 we have that

$$
\begin{aligned}
&F^2(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z^{\kappa,\mu}_{\mathcal{B}_2,\nu})\\
&= \frac{1}{C(\alpha;\gamma;\mathcal{B}_2)}\left\{\gamma_1\sum_{g=0}^{\infty}\frac{1}{\left[B_g^{(1)}\right]^{\alpha_1}}\sum_{k=B_g^{(1)}}^{B_{g+1}^{(1)}-1}\left|\frac{1}{B_\nu}\sum_{i=0}^{B_\nu-1}{}_{B_1}\mathrm{Vil}_k\big(\eta_{B_1}(i)\big)\right|^2\right.\\
&\quad+\gamma_2\sum_{g=0}^{\infty}\frac{1}{[B_g^{(2)}]^{\alpha_2}}\sum_{l=B_g^{(2)}}^{B_{g+1}^{(2)}-1}\left|\frac{1}{B_\nu}\sum_{i=0}^{B_\nu-1}{}_{B_2}\mathrm{Vil}_l\big(p_{B_2}(i)\big)\right|^2\\
&\quad+\gamma_1\gamma_2\left[\sum_{g_1=0}^{\nu-1}\sum_{g_2=0}^{\nu-1}+\sum_{g_1=0}^{\nu-1}\sum_{g_2=\nu}^{\infty}+\sum_{g_1=\nu}^{\infty}\sum_{g_2=0}^{\nu-1}+\sum_{g_1=\nu}^{\infty}\sum_{g_2=\nu}^{\infty}\right]\frac{1}{[B_{g_1}^{(1)}]^{\alpha_1}}\frac{1}{[B_{g_2}^{(2)}]^{\alpha_2}}\\
&\quad\left.\times\sum_{k_1=B_{g_1}^{(1)}}^{B_{g_1+1}^{(1)}-1}\sum_{k_2=B_{g_2}^{(2)}}^{B_{g_2+1}^{(2)}-1}\left|\frac{1}{B_\nu}\sum_{i=0}^{B_\nu-1}{}_{B_1}\mathrm{Vil}_{k_1}\big(\eta_{B_1}(i)\big).{}_{B_2}\mathrm{Vil}_{k_2}\big(p_{B_2}(i)\big)\right|^2\right\}\\
&= \frac{1}{C(\alpha;\gamma;\mathcal{B}_2)}[\gamma_1\Sigma_1+\gamma_2\Sigma_2+\gamma_1\gamma_2(\Sigma_3+\Sigma_4+\Sigma_5+\Sigma_6)],
\end{aligned}
\tag{7}
$$

where the constant

$$
C(\alpha;\gamma;\mathcal{B}_2)=\mu(\alpha_1;\gamma_1;B_1)+\mu(\alpha_2;\gamma_2;B_2)+\mu(\alpha_1;\gamma_1;B_1)\cdot\mu(\alpha_2;\gamma_2;B_2).
$$

According to statements (i) and (ii) of Lemma 1 for the sum Σ_1 we consecutively obtain:

$$
\begin{aligned}
\Sigma_1 &= \sum_{g=\nu}^{\infty}\frac{1}{\left[B_g^{(1)}\right]^{\alpha_1}}\sum_{\substack{B_g^{(1)}\le k\le B_{g+1}^{(1)}-1\\ k\equiv 0 \pmod{B_\nu}}}1\\
&= \sum_{g=\nu}^{\infty}\frac{1}{\left[B_g^{(1)}\right]^{\alpha_1}}\cdot\left[b_\nu^{(1)}\cdot b_{\nu+1}^{(1)}\cdots b_{g-1}^{(1)}(b_g^{(1)}-1)\right]\le\frac{M-1}{B_\nu}\sum_{g=\nu}^{\infty}\frac{1}{\left[B_g^{(1)}\right]^{\alpha_1-1}}\\
&= \frac{M-1}{B_\nu}\left[\frac{1}{[B_\nu^{(1)}]^{\alpha_1-1}}+\frac{1}{[B_{\nu+1}^{(1)}]^{\alpha_1-1}}+\frac{1}{[B_{\nu+2}^{(1)}]^{\alpha_1-1}}+\cdots\right]\\
&= \frac{M-1}{B_\nu\cdot[B_\nu^{(1)}]^{\alpha_1-1}}\left[1+\frac{1}{[b_\nu^{(1)}]^{\alpha_1-1}}+\frac{1}{[b_\nu^{(1)}b_{\nu+1}^{(1)}]^{\alpha_1-1}}+\cdots\right]\\
&\le \frac{M-1}{B_\nu^{\alpha_1}}\left[1+\frac{1}{2^{\alpha_1-1}}+\frac{1}{(2^{\alpha_1-1})^2}+\cdots\right]=\frac{(M-1)\cdot 2^{\alpha_1}}{2^{\alpha_1}-2}\frac{1}{B_\nu^{\alpha_1}};
\end{aligned}
\tag{8}
$$

$$\begin{aligned}
\Sigma_1 &\geq \sum_{g=\nu}^{\infty} \frac{1}{\left[B_g^{(1)}\right]^{\alpha_1}} \left[b_\nu^{(1)} \cdot b_{\nu+1}^{(1)} \cdots b_{g-1}^{(1)}\right] \\
&= \frac{1}{B_\nu} \sum_{g=\nu}^{\infty} \frac{1}{\left[B_g^{(1)}\right]^{\alpha_1-1}} \\
&= \frac{1}{B_\nu} \frac{1}{[B_\nu^{(1)}]^{\alpha_1-1}} \left[1 + \frac{1}{[b_{\nu+1}^{(1)}]^{\alpha_1-1}} + \frac{1}{[b_{\nu+1}^{(1)} b_{\nu+2}^{(1)}]^{\alpha_1-1}} + \cdots\right] \\
&\geq \frac{1}{B_\nu^{\alpha_1}} \left[1 + \frac{1}{M^{\alpha_1-1}} + \frac{1}{(M^{\alpha_1-1})^2} + \cdots\right] \\
&= \frac{M^{\alpha_1}}{M^{\alpha_1} - M} \frac{1}{B_\nu^{\alpha_1}}.
\end{aligned} \tag{9}$$

By using similar technique we can prove that

$$\frac{M^{\alpha_2}}{M^{\alpha_2} - M} \frac{1}{B_\nu^{\alpha_2}} \leq \Sigma_2 \leq \frac{(M-1)\cdot 2^{\alpha_2}}{2^{\alpha_2} - 2} \frac{1}{B_\nu^{\alpha_2}}. \tag{10}$$

Now, we will estimate the sum Σ_3. For this purpose we will use the statement (i) of Lemma 2. By using the introduced sets $A(g_1;g)$ we consecutively obtain

$$\begin{aligned}
\Sigma_3 &= \sum_{g_1=0}^{\nu-1} \frac{1}{[B_{g_1}^{(1)}]^{\alpha_1}} \sum_{g=0}^{g_1} \sum_{k_1\in A(g_1;g)} \sum_{g_2=0}^{\nu-1} \frac{1}{[B_{g_2}^{(2)}]^{\alpha_2}} \\
&\times \sum_{k_2=B_{g_2}^{(2)}}^{B_{g_2+1}^{(2)}-1} \left| \frac{1}{B_\nu} \sum_{i=0}^{B_\nu-1} {}_{\mathcal{B}_1}\mathrm{Vil}_{k_1}\big(\eta_{B_1}(i)\big) \cdot {}_{\mathcal{B}_2}\mathrm{Vil}_{k_2}\big(p_{B_2}(i)\big) \right|^2 \\
&= \sum_{g_1=0}^{\nu-1} \frac{1}{[B_{g_1}^{(1)}]^{\alpha_1}} \sum_{g=0}^{g_1} \frac{1}{[B_{\nu-1-g}^{(2)}]^{\alpha_2}} \sum_{k_1\in A(g_1;g)} 1.
\end{aligned} \tag{11}$$

For the cardinality of the set $A(g_1;g)$ we will use the bounds

$$b_{g+1}^{(1)} \cdots b_{g_1-1}^{(1)} \leq |A(g_1;g)| \leq b_g^{(1)} b_{g+1}^{(1)} \cdots b_{g_1-1}^{(1)} \cdot (M-1),$$

as we assume that in the cases when $g=g_1–1$ and $g=g_1$ we have $b_{g+1}^{(1)}\cdots b_{g_1-1}^{(1)}=1$, also if $g=g_1$, then $b_g^{(1)} b_{g+1}^{(1)} \cdots b_{g_1-1}^{(1)} = 1$. Hence, we obtain that

$$|A(g_1;g)| \geq \frac{B_{g_1}^{(1)}}{b_g^{(1)} \cdot B_g^{(1)}} \geq \frac{1}{M} \frac{B_{g_1}^{(1)}}{B_g^{(1)}}.$$

We directly can check up that the last inequality is true if $g = g_1 - 1$ and $g = g_1$. From (11) we consecutively obtain

$$\begin{aligned}
\Sigma_3 &\leq (M-1) \sum_{g_1=0}^{\nu-1} \frac{1}{[B^{(1)}_{g_1}]^{\alpha_1}} \sum_{g=0}^{g_1} \frac{1}{[B^{(2)}_{\nu-1-g}]^{\alpha_2}} \left[b^{(1)}_g b^{(1)}_{g+1} \cdots b^{(1)}_{g_1-1}\right] \\
&= (M-1) \sum_{g_1=0}^{\nu-1} \frac{1}{[B^{(1)}_{g_1}]^{\alpha_1-1}} \sum_{g=0}^{g_1} \frac{1}{B^{(1)}_g} \frac{1}{[B^{(2)}_{\nu-1-g}]^{\alpha_2}} \\
&\leq (M-1)M^{\alpha_2} \frac{1}{B^{\alpha_2}_\nu} \sum_{g_1=0}^{\nu-1} \frac{1}{[B^{(1)}_{g_1}]^{\alpha_1-1}} \sum_{g=0}^{g_1} \left[B^{(1)}_g\right]^{\alpha_2-1} \\
&\leq (M-1)M^{\alpha_2} \frac{1}{B^{\alpha_2}_\nu} \sum_{g_1=0}^{\nu-1} \frac{1}{[B^{(1)}_{g_1}]^{\alpha_1-1}} \\
&\qquad\qquad [B^{(1)}_{g_1}]^{\alpha_2-1} \left[1 + \frac{1}{2^{\alpha_2-1}} + \frac{1}{(2^{\alpha_2-1})^2} + \cdots\right] \\
&= \frac{(M-1)(2M)^{\alpha_2}}{2^{\alpha_2}-2} \frac{1}{B^{\alpha_2}_\nu} \sum_{g=0}^{\nu-1} [B^{(1)}_g]^{\alpha_2-\alpha_1};
\end{aligned} \tag{12}$$

$$\begin{aligned}
\Sigma_3 &\geq \frac{1}{M} \sum_{g_1=0}^{\nu-1} \frac{1}{[B^{(1)}_{g_1}]^{\alpha_1-1}} \sum_{g=0}^{g_1-1} \frac{1}{B^{(1)}_g} \frac{1}{[B^{(2)}_{\nu-1-g}]^{\alpha_2}} \\
&\geq \frac{2^{\alpha_2}}{M} \frac{1}{B^{\alpha_2}_\nu} \sum_{g_1=0}^{\nu-1} \frac{1}{[B^{(1)}_{g_1}]^{\alpha_1-1}} \sum_{g=0}^{g_1-1} [B^{(1)}_g]^{\alpha_2-1} \\
&> \frac{2^{\alpha_2}}{M} \frac{1}{B^{\alpha_2}_\nu} \sum_{g_1=0}^{\nu-1} \frac{1}{[B^{(1)}_{g_1}]^{\alpha_1-1}} [B^{(1)}_{g_1-1}]^{\alpha_2-1} \\
&\geq \frac{2^{\alpha_2}}{M \cdot M^{\alpha_2-1}} \frac{1}{B^{\alpha_2}_\nu} \sum_{g_1=0}^{\nu-1} \frac{1}{[B^{(1)}_{g_1}]^{\alpha_1-1}} \cdot [B^{(1)}_{g_1}]^{\alpha_2-1} \\
&= \frac{2^{\alpha_2}}{M^{\alpha_2}} \frac{1}{B^{\alpha_2}_\nu} \sum_{g=0}^{\nu-1} [B^{(1)}_g]^{\alpha_2-\alpha_1}.
\end{aligned} \tag{13}$$

To estimate the sum Σ_4 we will use the integers k_2'' introduced in the condition of the statement (ii) of Lemma 2. We will obtain the following:

$$\begin{aligned}
\Sigma_4 &= \sum_{g_1=0}^{\nu-1} \frac{1}{[B_{g_1}^{(1)}]^{\alpha_1}} \sum_{g_2=\nu}^{\infty} \frac{1}{[B_{g_2}^{(2)}]^{\alpha_2}} \sum_{k_1=B_{g_1}^{(1)}}^{B_{g_1+1}^{(1)}-1} \sum_{k_2''} 1 \\
&= \sum_{g_1=0}^{\nu-1} \frac{(b_{g_1}^{(1)}-1)B_{g_1}^{(1)}}{[B_{g_1}^{(1)}]^{\alpha_1}} \sum_{g_2=\nu}^{\infty} \frac{1}{[B_{g_2}^{(2)}]^{\alpha_2}} \left[b_\nu^{(2)} b_{\nu+1}^{(2)} \cdots b_{g_2-1}^{(2)}(b_{g_2}^{(2)}-1)\right] \\
&\le (M-1)^2 \sum_{g_1=0}^{\nu-1} \frac{1}{[B_{g_1}^{(1)}]^{\alpha_1-1}} \sum_{g_2=\nu}^{\infty} \frac{1}{[B_{g_2}^{(2)}]^{\alpha_2}} \left[b_\nu^{(2)} b_{\nu+1}^{(2)} \cdots b_{g_2-1}^{(2)}\right] \\
&= (M-1)^2 \frac{1}{B_\nu} \sum_{g_1=0}^{\nu-1} \frac{1}{[B_{g_1}^{(1)}]^{\alpha_1-1}} \sum_{g_2=\nu}^{\infty} \frac{1}{[B_{g_2}^{(2)}]^{\alpha_2-1}} \qquad (14)\\
&< (M-1)^2 \frac{1}{B_\nu} \sum_{g_1=0}^{\infty} \frac{1}{[B_{g_1}^{(1)}]^{\alpha_1-1}} \sum_{g_2=\nu}^{\infty} \frac{1}{[B_{g_2}^{(2)}]^{\alpha_2-1}} \\
&\le (M-1)^2 \frac{1}{B_\nu} \left[1 + \frac{1}{2^{\alpha_1-1}} + \frac{1}{(2^{\alpha_1-1})^2} + \cdots\right] \\
&\qquad\qquad \frac{1}{B_\nu^{\alpha_2-1}} \left[1 + \frac{1}{2^{\alpha_2-1}} + \frac{1}{(2^{\alpha_2-1})^2} + \cdots\right] \\
&= (M-1)^2 \frac{2^{\alpha_1+\alpha_2}}{(2^{\alpha_1}-2)(2^{\alpha_2}-2)} \frac{1}{B_\nu^{\alpha_2}};
\end{aligned}$$

$$\begin{aligned}
\Sigma_4 &\ge \sum_{g_1=0}^{\nu-1} \frac{1}{[B_{g_1}^{(1)}]^{\alpha_1-1}} \sum_{g_2=\nu}^{\infty} \frac{1}{[B_{g_2}]^{\alpha_2}} \left[b_\nu^{(2)} b_{\nu+1}^{(2)} \cdots b_{g_2-1}^{(2)}\right] \\
&= \frac{1}{B_\nu} \sum_{g_1=0}^{\nu-1} \frac{1}{[B_{g_1}^{(1)}]^{\alpha_1-1}} \sum_{g_2=\nu}^{\infty} \frac{1}{[B_{g_2}^{(2)}]^{\alpha_2-1}} \\
&\ge \frac{1}{B_\nu} \frac{1}{[B_0^{(1)}]^{\alpha_1-1}} \frac{1}{[B_\nu^{(2)}]^{\alpha_2-1}} \left[1 + \frac{1}{[b_\nu^{(2)}]^{\alpha_2-1}} + \frac{1}{[b_\nu^{(2)} \cdot b_{\nu+1}^{(2)}]^{\alpha_2-1}} + \cdots\right] \qquad (15)\\
&\ge \frac{1}{B_\nu^{\alpha_2}} \left[1 + \frac{1}{M^{\alpha_2-1}} + \frac{1}{(M^{\alpha_2-1})^2} + \cdots\right] \\
&= \frac{M^{\alpha_2}}{M^{\alpha_2}-M} \frac{1}{B_\nu^{\alpha_2}}.
\end{aligned}$$

By using the statements (iii) and (iv) of Lemma 2 we can prove the estimations

$$\frac{M^{\alpha_1}}{M^{\alpha_1}-M}\frac{1}{B_\nu^{\alpha_1}} \le \Sigma_5 < (M-1)^2\frac{2^{\alpha_1+\alpha_2}}{(2^{\alpha_1}-2)(2^{\alpha_2}-2)}\frac{1}{B_\nu^{\alpha_1}} \tag{16}$$

and

$$\frac{M^{\alpha_1+\alpha_2}}{(M^{\alpha_1}-M)(M^{\alpha_2}-M)}\frac{1}{B_\nu^{\alpha_1+\alpha_2-1}} \le \Sigma_6 \le (M-1)^2\frac{2^{\alpha_1+\alpha_2}}{(2^{\alpha_1}-2)(2^{\alpha_2}-2)}\frac{1}{B_\nu^{\alpha_1+\alpha_2-1}}. \tag{17}$$

From (7), (8), (9), (10), (12), (13), (14), (15), (16) and (17) we obtain that

$$\begin{aligned}
&F^2(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z^{\kappa,\mu}_{\mathcal{B}_2,\nu})\\
&\ge \frac{1}{C(\alpha;\gamma;\mathcal{B}_2)}\left\{\frac{2^{\alpha_2}}{M^{\alpha_2}}\frac{1}{B_\nu^{\alpha_2}}\sum_{g=0}^{\nu-1}\big[B_g^{(1)}\big]^{\alpha_2-\alpha_1}+\frac{2M^{\alpha_1}}{M^{\alpha_1}-1}\frac{1}{B_\nu^{\alpha_1}}\right.\\
&\quad\left.+\frac{2M^{\alpha_2}}{M^{\alpha_2}-1}\frac{1}{B_\nu^{\alpha_2}}+\frac{M^{\alpha_1+\alpha_2}}{(M^{\alpha_1}-M)(M^{\alpha_2}-M)}\frac{1}{B_\nu^{\alpha_1+\alpha_2-1}}\right\}\\
&= C'_1\frac{1}{B_\nu^{\alpha_2}}\sum_{g=0}^{\nu-1}\big[B_g^{(1)}\big]^{\alpha_2-\alpha_1}+C'_2\frac{1}{B_\nu^{\alpha_1}}+C'_3\frac{1}{B_\nu^{\alpha_2}}+C'_4\frac{1}{B_\nu^{\alpha_1+\alpha_2-1}}
\end{aligned}$$

and

$$\begin{aligned}
&F^2(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z^{\kappa,\mu}_{\mathcal{B}_2,\nu})\\
&< \frac{1}{C(\alpha;\gamma;\mathcal{B}_2)}\left\{\gamma_1\gamma_2\frac{(M-1)(2M)^{\alpha_2}}{2^{\alpha_2}-2}\frac{1}{B_\nu^{\alpha_2}}\sum_{g=0}^{\nu-1}\big[B_g^{(1)}\big]^{\alpha_2-\alpha_1}\right.\\
&\quad+(M-1)\frac{2^{\alpha_1}}{2^{\alpha_1}-2}\left(\gamma_1+\gamma_1\gamma_2(M-1)\frac{2^{\alpha_2}}{2^{\alpha_2}-2}\right)\frac{1}{B_\nu^{\alpha_1}}\\
&\quad+(M-1)\frac{2^{\alpha_2}}{2^{\alpha_2}-2}\left(\gamma_2+\gamma_1\gamma_2(M-1)\frac{2^{\alpha_1}}{2^{\alpha_1}-2}\right)\frac{1}{B_\nu^{\alpha_2}}\\
&\quad\left.+\gamma_1\gamma_2(M-1)^2\frac{2^{\alpha_1+\alpha_2}}{(2^{\alpha_1}-2)(2^{\alpha_2}-2)}\frac{1}{B_\nu^{\alpha_1+\alpha_2-1}}\right\}\\
&= C_1\frac{1}{B_\nu^{\alpha_2}}\sum_{g=0}^{\nu-1}\big[B_g^{(1)}\big]^{\alpha_2-\alpha_1}+C_2\frac{1}{B_\nu^{\alpha_1}}+C_3\frac{1}{B_\nu^{\alpha_2}}+C_4\frac{1}{B_\nu^{\alpha_1+\alpha_2-1}},
\end{aligned}$$

where the values of the constants C'_τ and C_τ, $\tau=1,2,3,4$, are evident. Theorem 1 is finally proved. $\square$

P r o o f o f T h e o r e m 2. According to results of Theorem 1 the following inequalities hold

$$\begin{aligned}
&C_1'\frac{1}{B_\nu^{\alpha_2}}\sum_{g=0}^{\nu-1}\big[B_g^{(1)}\big]^{\alpha_2-\alpha_1}+C_2'\frac{1}{B_\nu^{\alpha_1}}+C_3'\frac{1}{B_\nu^{\alpha_2}}+C_4'\frac{1}{B_\nu^{\alpha_1+\alpha_2-1}}\\
&\le F^2(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z_{\mathcal{B}_2,\nu}^{\kappa,\mu}) \qquad (18)\\
&< C_1\frac{1}{B_\nu^{\alpha_2}}\sum_{g=0}^{\nu-1}\big[B_g^{(1)}\big]^{\alpha_2-\alpha_1}+C_2\frac{1}{B_\nu^{\alpha_1}}+C_3\frac{1}{B_\nu^{\alpha_2}}+C_4\frac{1}{B_\nu^{\alpha_1+\alpha_2-1}}.
\end{aligned}$$

(i) Let us assume that $\alpha_1=\alpha_2$. Then, from (18) we obtain that

$$\begin{aligned}
&C_1'\frac{\nu}{B_\nu^{\alpha_2}}+(C_2'+C_3')\frac{1}{B_\nu^{\alpha_2}}+C_4'\frac{1}{B_\nu^{2\alpha_2-1}}\\
&\le F^2(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z_{\mathcal{B}_2,\nu}^{\kappa,\mu}) \qquad (19)\\
&< C_1\frac{\nu}{B_\nu^{\alpha_2}}+(C_2+C_3)\frac{1}{B_\nu^{\alpha_2}}+C_4\frac{1}{B_\nu^{2\alpha_2-1}}.
\end{aligned}$$

Let us denote $B_\nu=N$. Then, the equality $N=b_0^{(1)}\cdot b_1^{(1)}\cdots b_{\nu-1}^{(1)}$ gives us that

$$2^\nu\le N\le M^\nu,\qquad \frac{\log N}{\log M}\le\nu\le\frac{\log N}{\log 2}.$$

Hence, from (19) we obtain that

$$\begin{aligned}
&\frac{C_1'}{\log M}\frac{\log N}{N^{\alpha_2}}+(C_2'+C_3')\frac{1}{N^{\alpha_2}}+C_4'\frac{1}{N^{2\alpha_2-1}}\\
&\le F^2(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z_{\mathcal{B}_2,\nu}^{\kappa,\mu})<\frac{C_1}{\log 2}\frac{\log N}{N^{\alpha_2}}+(C_2+C_3)\frac{1}{N^{\alpha_2}}+C_4\frac{1}{N^{2\alpha_2-1}},
\end{aligned}$$

$$\begin{aligned}
&\lim_{N\to\infty}\left\{\frac{C_1'}{\log M}\frac{\log N}{N^{\alpha_2}}+(C_2'+C_3')\frac{1}{N^{\alpha_2}}+C_4'\frac{1}{N^{2\alpha_2-1}}\right\}\\
&\le\lim_{\nu\to\infty}F^2(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z_{\mathcal{B}_2,\nu}^{\kappa,\mu})\\
&\le\lim_{N\to\infty}\left\{\frac{C_1}{\log 2}\frac{\log N}{N^{\alpha_2}}+(C_2+C_3)\frac{1}{N^{\alpha_2}}+C_4\frac{1}{N^{2\alpha_2-1}}\right\},
\end{aligned}$$

so

$$\frac{C_1'}{\log M}\le\lim_{N\to\infty}\frac{N^{\alpha_2}\cdot F^2(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z_{\mathcal{B}_2,\nu}^{\kappa,\mu})}{\log N}\le\frac{C_1}{\log 2}$$

and

$$F(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z_{\mathcal{B}_2,\nu}^{\kappa,\mu})=\mathcal{O}\Big(\frac{\sqrt{\log N}}{N^{\frac{\alpha_2}{2}}}\Big).$$

The order $\mathcal{O}\big(\frac{\sqrt{\log N}}{N^{\frac{\alpha_2}{2}}}\big)$ is the exact order of the diaphony $F(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z^{\kappa,\mu}_{\mathcal{B}_2,\nu})$.

(i1) If $1<\alpha_2<2$, then there exists some $\varepsilon>0$ such that $\frac{\alpha_2}{2}=1-\varepsilon$ and from (i) follows that $F(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z^{\kappa,\mu}_{\mathcal{B}_2,\nu})=\mathcal{O}\left(\frac{\sqrt{\log N}}{N^{1-\varepsilon}}\right)$.

(i2) If $\alpha_2=2$, then from (i) $F(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z^{\kappa,\mu}_{\mathcal{B}_2,\nu})=\mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right)$.

(i3) If $\alpha_2>2$, then there exists some $\varepsilon>0$ such that $\frac{\alpha_2}{2}=1+\varepsilon$ and from (i) follows that $F(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z^{\kappa,\mu}_{\mathcal{B}_2,\nu})=\mathcal{O}\left(\frac{\sqrt{\log N}}{N^{1+\varepsilon}}\right)$.

(ii) Let us assume that $\alpha_1>\alpha_2$. Then, the inequalities hold

$$\begin{aligned}1<\sum_{g=0}^{\nu-1}\big[B_g^{(1)}\big]^{\alpha_2-\alpha_1}&<\sum_{g=0}^{\infty}\big[B_g^{(1)}\big]^{\alpha_2-\alpha_1}\\&=1+\frac{1}{[b_0^{(1)}]^{\alpha_1-\alpha_2}}+\frac{1}{[b_0^{(1)}b_1^{(1)}]^{\alpha_1-\alpha_2}}+\cdots\\&\le 1+\frac{1}{2^{\alpha_1-\alpha_2}}+\frac{1}{(2^{\alpha_1-\alpha_2})^2}+\cdots=\frac{2^{\alpha_1}}{2^{\alpha_1}-2^{\alpha_2}}.\end{aligned}$$

From (18) and the above inequalities we obtain

$$\begin{aligned}(C_1'+C_3')\frac{1}{B_\nu^{\alpha_2}}+C_2'\frac{1}{B_\nu^{\alpha_1}}+C_4'\frac{1}{B_\nu^{\alpha_1+\alpha_2-1}}&<F^2(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z^{\kappa,\mu}_{\mathcal{B}_2,\nu})\\&<\left(\frac{C_1\cdot 2^{\alpha_1}}{2^{\alpha_1}-2^{\alpha_2}}+C_3\right)\frac{1}{B_\nu^{\alpha_2}}+C_2\frac{1}{B_\nu^{\alpha_1}}+C_4\frac{1}{B_\nu^{\alpha_1+\alpha_2-1}},\end{aligned}$$

$$\begin{aligned}C_1'+C_3'+C_2'\frac{1}{B_\nu^{\alpha_1-\alpha_2}}+C_4'\frac{1}{B_\nu^{\alpha_1-1}}&<B_\nu^{\alpha_2}\cdot F^2(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z^{\kappa,\mu}_{\mathcal{B}_2,\nu})\\&<\frac{C_1\cdot 2^{\alpha_1}}{2^{\alpha_1}-2^{\alpha_2}}+C_3+C_2\frac{1}{B_\nu^{\alpha_1-\alpha_2}}+C_4\frac{1}{B_\nu^{\alpha_1-1}},\end{aligned}$$

$$\begin{aligned}&\lim_{N\to\infty}\left[C_1'+C_3'+C_2'\frac{1}{N^{\alpha_1-\alpha_2}}+C_4'\frac{1}{N^{\alpha_1-1}}\right]\\&\qquad\le\lim_{N\to\infty}N^{\alpha_2}\cdot F^2(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z^{\kappa,\mu}_{\mathcal{B}_2,\nu})\\&\qquad\qquad\le\lim_{N\to\infty}\left[\frac{C_1\cdot 2^{\alpha_1}}{2^{\alpha_1}-2^{\alpha_2}}+C_3+\frac{C_2}{N^{\alpha_1-\alpha_2}}+\frac{C_4}{N^{\alpha_1-1}}\right],\end{aligned}$$

$$C_1'+C_3'\le\lim_{N\to\infty}N^{\alpha_2}\cdot F^2(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z^{\kappa,\mu}_{\mathcal{B}_2,\nu})\le\frac{C_1\cdot 2^{\alpha_1}}{2^{\alpha_1}-2^{\alpha_2}}+C_3$$

and

$$F(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z^{\kappa,\mu}_{\mathcal{B}_2,\nu})=\mathcal{O}\left(\frac{1}{N^{\frac{\alpha_2}{2}}}\right).$$

It is clear that the order $\mathcal{O}\Big(\frac{1}{N^{\frac{\alpha_2}{2}}}\Big)$ is the exact order of the diaphony $F(Vil_{\mathcal{B}_2};\alpha;\gamma;Z^{\kappa,\mu}_{\mathcal{B}_2,\nu})$.

(ii1) If $1<\alpha_2<2$, then there exists some $\varepsilon>0$ such that $\frac{\alpha_2}{2}=1-\varepsilon$ and from (ii) follows that $F(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z^{\kappa,\mu}_{\mathcal{B}_2,\nu})=\mathcal{O}\left(\frac{1}{N^{1-\varepsilon}}\right)$.

(ii2) If $\alpha_2=2$, then from (ii) follows that $F(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z^{\kappa,\mu}_{\mathcal{B}_2,\nu})=\mathcal{O}\left(\frac{1}{N}\right)$.

(ii3) If $\alpha_2>2$, then there exists some $\varepsilon>0$ such that $\frac{\alpha_2}{2}=1+\varepsilon$ and from (ii) follows that $F(\mathrm{Vil}_{\mathcal{B}_2};\alpha;\gamma;Z^{\kappa,\mu}_{\mathcal{B}_2,\nu})=\mathcal{O}\left(\frac{1}{N^{1+\varepsilon}}\right)$.

Theorem 2 is finally proved. □

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