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Vassil Grozdanov
South-West University "Neofit Rilski"

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THE WEIGHTED DIAPHONY

V. Ristovska, V. Grozdanov

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Abstract

In the present paper the authors introduce a new quantitative measure for uniform distribution of sequences in $[0,1)^s$, the so-called "weighted diaphony." The definition of the weighted diaphony is based on using the trigonometric functional system. At a special choice of the parameters α and $\boldsymbol{\gamma}$ on which the weighted diaphony depends, the "classical" diaphony introduced by Zinterhof is obtained. It is shown that the computing complexity of the weighted diaphony of an arbitrary net composed of N points in $[0,1)^s$ is $\mathcal{O}(S.N^2)$. Relationship between the worst-case error of the quasi-Monte Carlo integration in a class of weighted Hilbert space and the weighted diaphony is obtained.
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1. Introduction. Let $s \geq 1$ be a fixed integer and $\xi = (\mathbf{x}_n)_{n\geq 0}$ is an arbitrary sequence of points in the s-dimensional unit cube $[0,1)^s$. For an arbitrary subinterval $J \subseteq [0,1)^s$ with a volume $\mu(J)$ and an arbitrary integer $N \geq 1$ we denote with $A_N(\xi; J)$ the number of the points of the sequence ξ which indices n satisfy $0 \leq n \leq N-1$ and belong to the interval J. The sequence ξ is called uniformly distributed if the equality $\lim_{N\to\infty} N^{-1} A_N(\xi; J) = \mu(J)$ holds for every subinterval J of $[0,1)^s$.

The examination of numerical measures for distribution of sequences in $[0,1)^s$ is of interest of the quantitative theory of the uniform distribution of sequences. In general the discrepancy and the diaphony are quantitative measures for the irregularity of the distribution of sequences in $[0,1)^s$.

Let $\mathcal{T} = \{\exp(2\pi\mathbf{i}\langle\mathbf{m},\mathbf{x}\rangle) : \mathbf{m} = (m_1,\ldots,m_s) \in \mathbf{Z}^s,\ \mathbf{x} = (x_1,\ldots,x_s) \in [0,1)^s,\ \langle\mathbf{m},\mathbf{x}\rangle = \sum_{j=1}^s m_j x_j\}$ be the trigonometric functional system. Zinterhof [1] introduces the "classical" diaphony as a numerical measure for distribution of sequences in $[0,1)^s$. For each integer $N \geq 1$ the diaphony $F_N(\mathcal{T};\xi)$ of the first N elements of the sequence $\xi = (\mathbf{x}_n)_{n\geq 0}$ in $[0,1)^s$ is defined as

$$F_N(\mathcal{T};\xi) = \left(\sum_{\mathbf{m}\in\mathbf{Z}^s, \mathbf{m}\neq\mathbf{0}} R^{-2}(\mathbf{m}) \left| \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi\mathbf{i}\langle\mathbf{m},\mathbf{x}_n\rangle) \right|^2 \right)^{\frac{1}{2}},$$

where for each vector $\mathbf{m} = (m_1, \dots, m_s) \in \mathbf{Z}^s$ $R(\mathbf{m}) = \prod_{j=1}^{s} \max(1, |m_j|)$.

For multivariate numerical integration Sloan and Wożniakowski [2] propose to arrange the coordinates $x_1, x_2, \dots, x_s$ of the functions which are integrated and respectively the coordinates of the nets which we use in the process of the numerical integration in such a way that x_1 is the most important coordinate, x_2 is the next one, and so on. This is realized by associating nonincreasing weights $\gamma_1, \gamma_2, \dots, \gamma_s$ to the successive coordinate direction. Following Sloan and Wożniakowski we will recall the concept for the weighted $\mathcal{L}_2$ discrepancy. For an arbitrary sequence ξ of points in $[0,1)^s$ and an arbitrary integer $N \geq 1$, define the discrepancy function as $\Delta_N(\xi; \mathbf{t}) = N^{-1} A_N(\xi; [0,t_1) \times \cdots \times [0,t_s)) - t_1 \dots t_s$, where for $1 \leq j \leq s$ $t_j \in [0,1)$ and $\mathbf{t} = (t_1, \dots, t_s)$.

Let D denote the index set $D = \{1, 2, \dots, s\}$. For an arbitrary subset $u \subseteq D$ let $|u|$ be the cardinality of u. For a vector $\mathbf{x} = (x_1, x_2, \dots, x_s) \in [0,1)^s$ let $\mathbf{x}_u$ denote the vector from $[0,1)^{|u|}$ containing all components of $\mathbf{x}$, whose indices are in u. We denote $d\mathbf{x}_u = \prod_{j \in u} dx_j$. Let $(\mathbf{x}_u, \mathbf{1})$ be the vector from $[0,1)^s$ with components x_j if $j \in u$ and components 1 if $j \notin u$.

For a given vector of weights $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_s)$, where $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_s > 0$ and nonempty $u \subseteq D$ we put $\boldsymbol{\gamma}_u = \prod_{j \in u} \gamma_j$. Then the weighted $\mathcal{L}_2$ discrepancy of the first N elements of the sequence ξ is defined as

$$\mathcal{L}_{2,\boldsymbol{\gamma},N}(\xi) = \left(\sum_{\emptyset \neq u \subseteq D} \boldsymbol{\gamma}_u \int_{[0,1)^{|u|}} \Delta_N^2(\xi; (\mathbf{x}_u, \mathbf{1})) d\mathbf{x}_u \right)^{\frac{1}{2}}.$$

The aims of this research are:

- To define a new version of the diaphony, the so-called "weighted diaphony" and to prove that the weighted diaphony is a quantitative measure for uniform distribution of sequences in $[0,1)^s$.
- To show that from the weighted diaphony definition at a special choice of the parameters on which it depends the "classical" diaphony is obtained.
- To show that the computing complexity of the weighted diaphony of an arbitrary net, composed of N points in $[0,1)^s$, is $\mathcal{O}(S.N^2)$.
- To obtain the relationship between the worst-case error of the integration in the weighted Hilbert space $H_{s,\alpha,\boldsymbol{\gamma}}$ (see Section 3) which is based on using determined nets in $[0,1)^s$ and the weighted diaphony of these nets.

2. Statement of the results. In order to give the form of the so-called "weighted diaphony" (see Definition 1), the parameters α and $\boldsymbol{\gamma}$ will be used. To characterize the importance of the different coordinates of the nets in $[0,1)^s$ the parameters (or "weights") $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_s)$, where $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_s > 0$ will be used. The fixed real number $\alpha > 1$ will characterize the rate of decay of the Fourier coefficients of a special function $G(\alpha, \boldsymbol{\gamma}; \cdot)$ constructed in Theorem 2. This function will play an important role in defining the weighted diaphony. In the next definition we will give the concept of the weighted diaphony:

Definition 1. Let $\alpha \geq 2$ be an arbitrary even integer and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_s)$, where $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_s > 0$ is an arbitrary vector of real weights. For each integer

$N \geq 1$ the weighted diaphony $F_N(\mathcal{T}; \alpha, \boldsymbol{\gamma}; \xi)$ of the first N elements of the sequence $\xi = (\mathbf{x}_n)_{n\geq 0}$ of points in $[0,1)^s$ is defined as

$$F_N(\mathcal{T}; \alpha, \boldsymbol{\gamma}; \xi)$$

$$= \left(\frac{1}{\prod_{j=1}^{s}[1 + 2\gamma_j \zeta(\alpha)] - 1} \sum_{\mathbf{m}\in \mathbf{Z}^s, \mathbf{m}\neq \mathbf{0}} r(\alpha, \boldsymbol{\gamma}; \mathbf{m}) \left| \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi \mathbf{i} \langle \mathbf{m}, \mathbf{x}_n \rangle) \right|^2 \right)^{\frac{1}{2}},$$

where for each vector $\mathbf{m} = (m_1, \ldots, m_s) \in \mathbf{Z}^s$ the coefficient $r(\alpha, \boldsymbol{\gamma}; \mathbf{m})$ is defined as $r(\alpha, \boldsymbol{\gamma}; \mathbf{m}) = \prod_{j=1}^{s} r(\alpha, \gamma_j; m_j)$ and for an arbitrary real $\gamma > 0$ and integer m we define

$$r(\alpha, \gamma; m) = \begin{cases} 1, & \text{if } m = 0, \\ \frac{\gamma}{m^\alpha}, & \text{if } m \neq 0, \end{cases}$$

and $\zeta(\alpha)$ is the well-known Riemann's zeta-function.

Theorem 1. The sequence ξ of points in $[0,1)^s$ is uniformly distributed if and only if the equality

$$\lim_{N\to\infty} F_N(\mathcal{T}; \alpha, \boldsymbol{\gamma}; \xi) = 0$$

holds for each even integer $\alpha \geq 2$ and for an arbitrary vector $\boldsymbol{\gamma}$ of positive weights.

From the point of view of the practical application of the weighted diaphony we note the following: for each integer $N \geq 1$ and an arbitrary sequence ξ of points in $[0,1)^s$ the inequalities

$$0 \leq F_N(\mathcal{T}; \alpha, \boldsymbol{\gamma}; \xi) \leq 1 \tag{1}$$

have to hold for each even integer $\alpha \geq 2$ and an arbitrary vector $\boldsymbol{\gamma}$ of positive weights. The multiplier $(\prod_{j=1}^{s}[1 + 2\gamma_j\zeta(\alpha)] - 1)^{-1}$ guarantees that the inequalities (1) hold.

For each integer $N \geq 1$ and an arbitrary sequence ξ of points in $[0,1)^s$ from the Definition 1 at the special choice of the parameters $\alpha = 2$ and $\boldsymbol{\gamma} = \mathbf{1} = (\mathbf{1}, \ldots, \mathbf{1})$ we obtain the equality $\sqrt{\left(1 + \frac{\pi^2}{3}\right)^s - 1} \cdot F_N(\mathcal{T}; 2, \mathbf{1}; \xi) = F_N(\mathcal{T}; \xi)$, i.e. the "classical" diaphony $F_N(\mathcal{T}; \xi)$ of the sequence ξ with an exactness of the multiplier $\sqrt{\left(1 + \frac{\pi^2}{3}\right)^s - 1}$ is obtained from Definition 1.

Lemma 1. For arbitrary integer numbers μ and h such that $0 \leq h \leq \mu$ let us define the coefficients $\varkappa(\mu, h) = -\frac{\mu!}{(\mu+1-h)!}$. Let $\alpha \geq 2$ be an arbitrary and fixed integer and let $\gamma > 0$ be an arbitrary real. We introduce the matrices

$$K(\alpha) = \begin{bmatrix} \varkappa(0,0) & \varkappa(1,0) & \varkappa(2,0) & \ldots & \varkappa(\alpha-1,0) & \varkappa(\alpha,0) \\ 0 & \varkappa(1,1) & \varkappa(2,1) & \ldots & \varkappa(\alpha-1,1) & \varkappa(\alpha,1) \\ 0 & 0 & \varkappa(2,2) & \ldots & \varkappa(\alpha-1,2) & \varkappa(\alpha,2) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \varkappa(\alpha-1,\alpha-1) & \varkappa(\alpha,\alpha-1) \\ 0 & 0 & 0 & \ldots & 0 & \varkappa(\alpha,\alpha) \end{bmatrix}$$

and $B(\alpha, \gamma) = [-1, 0, \ldots, 0, \gamma\cdot(2\pi\mathbf{i})^\alpha]^T$, where the matrix $B(\alpha, \gamma)$ is with $\alpha+1$ elements.

(i) We define the functions $\phi(\alpha, j)$ as: $\phi(\alpha,\alpha) = \frac{-1}{\alpha!}$ and for $j = \alpha-1, \alpha-2, \dots, 1, 0$ $\phi(\alpha, j) = \frac{1}{j!}\sum_{h=j+1}^{\alpha} \varkappa(h, j)\phi(\alpha, h)$. Then the solution $C = [C_0, C_1, \dots, C_\alpha]^T$ of the system $K(\alpha) \cdot C = B(\alpha, \gamma)$ is given in the form $C_0 = 1 + \phi(\alpha, 0)\gamma(2\pi\mathbf{i})^\alpha$ and for $1 \le j \le \alpha$ $C_j = \phi(\alpha, j)\gamma(2\pi\mathbf{i})^\alpha$.

(ii) For the solution of the system $K(\alpha) \cdot C = B(\alpha, \gamma)$ we obtain some partial results: $C_\alpha = -\frac{(2\pi\mathbf{i})^\alpha\gamma}{\alpha!}$, $C_{\alpha-1} = \frac{(2\pi\mathbf{i})^\alpha\gamma}{(\alpha-1)!2}$, $C_{\alpha-2} = -\frac{(2\pi\mathbf{i})^\alpha\gamma}{(\alpha-2)!12}$, $C_{\alpha-3} = 0$, $C_{\alpha-4} = \frac{(2\pi\mathbf{i})^\alpha\gamma}{(\alpha-4)!720}$, $C_{\alpha-5} = 0$, $C_{\alpha-6} = -\frac{(2\pi\mathbf{i})^\alpha\gamma}{(\alpha-6)!}\frac{1}{6.7!}$, $C_{\alpha-7} = 0$, $C_{\alpha-8} = \frac{(2\pi\mathbf{i})^\alpha\gamma}{(\alpha-8)!}\frac{3}{10!}$, $C_{\alpha-9} = 0$, $C_{\alpha-10} = -\frac{(2\pi\mathbf{i})^\alpha\gamma}{(\alpha-10)!}\frac{5}{6.11!}$, $C_{\alpha-11} = 0$, $C_{\alpha-12} = \frac{(2\pi\mathbf{i})^\alpha\gamma}{(\alpha-12)!}\frac{691}{15.14!}$, $C_{\alpha-13} = 0$.

Lemma 2. Let $\alpha \ge 2$ be an arbitrary fixed even integer.

(i) For an arbitrary real $\gamma > 0$ let us construct the polynomial

$$g(\alpha, \gamma; x) = C_\alpha x^\alpha + C_{\alpha-1}x^{\alpha-1} + \cdots + C_1 x + C_0, \quad x \in [0, 1),$$

where $C = [C_0, C_1, \dots, C_\alpha]^T$ is the solution of the system $K(\alpha) \cdot C = B(\alpha, \gamma)$. Then for each integer m the Fourier coefficient $\widehat{g}(\alpha, \gamma; m)$ of the polynomial $g(\alpha, \gamma; \cdot)$ satisfies the equality $\widehat{g}(\alpha, \gamma; m) = r(\alpha, \gamma; m)$, where the coefficient $r(\cdot)$ is defined in Definition 1.

(ii) Let $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_s)$, where $\gamma_1 \ge \gamma_2 \ge \cdots \ge \gamma_s > 0$ is an arbitrary vector of weights. For $1 \le j \le s$ let $B(\alpha, \gamma_j) = [-1, 0, \dots, 0, \gamma_j \cdot (2\pi\mathbf{i})^\alpha]^T$ and $C_j = [C_0^{(j)}, C_1^{(j)}, \dots, C_\alpha^{(j)}]^T$ are the solutions of the systems $K(\alpha) \cdot C_j = B(\alpha, \gamma_j)$. For $1 \le j \le s$ we define the polynomials $g(\alpha, \gamma_j; x) = C_\alpha^{(j)}x^\alpha + C_{\alpha-1}^{(j)}x^{\alpha-1} + \cdots + C_1^{(j)}x + C_0^{(j)}$, $x \in [0, 1)$ and

$$G(\alpha, \boldsymbol{\gamma}; \mathbf{x}) = -1 + \prod_{j=1}^{s} g(\alpha, \gamma_j; x_j), \quad \mathbf{x} = (x_1, \dots, x_s) \in [0, 1)^s.$$

Then for each vector $\mathbf{m} \in \mathbf{Z}^s$ the Fourier coefficient $\widehat{G}(\alpha, \boldsymbol{\gamma}; \mathbf{m})$ of the function $G(\alpha, \boldsymbol{\gamma}; \cdot)$ satisfies the equality

$$\widehat{G}(\alpha, \boldsymbol{\gamma}; \mathbf{m}) = \begin{cases} 0, & \text{if } \mathbf{m} = \mathbf{0}, \\ r(\alpha, \boldsymbol{\gamma}; \mathbf{m}), & \text{if } \mathbf{m} \ne \mathbf{0}. \end{cases} \tag{2}$$

Theorem 2. Let $N \ge 1$ be an arbitrary integer and $\xi_N = \{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1}\}$ is an arbitrary net, composed of N points in $[0, 1)^s$. For each even integer $\alpha \ge 2$ and an arbitrary vector of positive weights $\boldsymbol{\gamma}$ the weighted diaphony $F(\mathcal{T}; \alpha, \boldsymbol{\gamma}; \xi_N)$ of the net ξ_N satisfies the equality

$$F^2(\mathcal{T}; \alpha, \boldsymbol{\gamma}; \xi_N) = \frac{1}{\prod_{j=1}^{s}[1 + 2\gamma_j\zeta(\alpha)] - 1}\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} G(\alpha, \boldsymbol{\gamma}; \mathbf{x}_n - \mathbf{x}_m),$$

where the function $G(\cdot)$ is defined in the condition of Lemma 2.

3. Multivariate integration in the weighted Hilbert space $H_{s,\alpha,\boldsymbol{\gamma}}$. Let $\alpha > 1$ be an arbitrary real and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_s)$, where $\gamma_1 \ge \gamma_2 \ge \cdots \ge \gamma_s > 0$ is an arbitrary vector of weights. Sloan and Woźniakowski [3] introduce the reproducing kernel weighted Hilbert space $H_{s,\alpha,\boldsymbol{\gamma}}$ with a norm $\|\cdot\|_{s,\alpha,\boldsymbol{\gamma}}$. The space $H_{s,\alpha,\boldsymbol{\gamma}}$ is composed of 1-periodical with respect to all arguments complex-valued $L_1([0,1)^s)$ functions f, defined on $[0,1)^s$ with absolutely convergent Fourier series and for which $\|f\|_{s,\alpha,\boldsymbol{\gamma}} < \infty$.

Let for an arbitrary integer $N \geq 1$ $\xi_N = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1}\}$ be a net composed of N points in $[0,1)^s$. The worst-case error $e(\mathcal{T}; s, \alpha, \boldsymbol{\gamma}; \xi_N)$ of the integration in the space $H_{s,\alpha,\boldsymbol{\gamma}}$ is defined as

$$e(\mathcal{T}; s, \alpha, \boldsymbol{\gamma}; \xi_N) = \sup_{f \in H_{s,\alpha,\boldsymbol{\gamma}}, \|f\|_{s,\alpha,\boldsymbol{\gamma}} \leq 1} \left\{ \left| \int_{[0,1)^s} f(\mathbf{x})\, d\mathbf{x} - \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) \right| \right\}.$$

The next result is obtained.

Theorem 3. The worst-case error $e(\mathcal{T}; s, \alpha, \boldsymbol{\gamma}; \xi_N)$ of the integration in the reproducing kernel weighted Hilbert space $H_{s,\alpha,\boldsymbol{\gamma}}$, which is based on using the net ξ_N and the weighted diaphony of the net ξ_N are connected with the equality

$$e(\mathcal{T}; s, \alpha, \boldsymbol{\gamma}; \xi_N) = \sqrt{\prod_{j=1}^{s} [1 + 2\gamma_j \zeta(\alpha)] - 1} \cdot F(\mathcal{T}; \alpha, \boldsymbol{\gamma}; \xi_N).$$

4. Proof of the main results. Proof of Theorem 1. The Weyl criterion (see KUIPERS and NIEDERREITER [4], Theorem 6.2) gives that the sequence $\xi = (\mathbf{x}_n)_{n \geq 0}$ is uniformly distributed in $[0,1)^s$ if and only if the equality

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi \mathbf{i} \langle \mathbf{m}, \mathbf{x}_n \rangle) = 0 \tag{3}$$

holds for each vector $\mathbf{m} \in \mathbf{Z}^s$ and $\mathbf{m} \neq \mathbf{0}$.

Let the sequence ξ be uniformly distributed in $[0,1)^s$. From (3) for every $\epsilon > 0$ there is an index N_ϵ such that for each integer $N \geq N_\epsilon$ the inequality

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi \mathbf{i} \langle \mathbf{m}, \mathbf{x}_n \rangle) \right| < \epsilon \tag{4}$$

holds. From Definition 1 and (4) we have that for each even integer $\alpha \geq 2$ and for an arbitrary vector $\boldsymbol{\gamma}$ of positive weights

$$F_N^2(\mathcal{T}; \alpha, \boldsymbol{\gamma}; \xi) < \frac{\epsilon^2}{\prod_{j=1}^{s} [1 + 2\gamma_j \zeta(\alpha)] - 1} \sum_{\mathbf{m} \in \mathbf{Z}^s, \mathbf{m} \neq \mathbf{0}} r(\alpha, \boldsymbol{\gamma}; \mathbf{m}) = \epsilon^2.$$

The last inequality gives that the equality $\lim_{N \to \infty} F_N(\mathcal{T}; \alpha, \boldsymbol{\gamma}; \xi) = 0$ holds for each even integer $\alpha \geq 2$ and an arbitrary vector $\boldsymbol{\gamma}$ of positive weights.

We note that for each fixed vector $\mathbf{m} \in \mathbf{Z}^s$ and $\mathbf{m} \neq \mathbf{0}$ we have that

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi \mathbf{i} \langle \mathbf{m}, \mathbf{x}_n \rangle) \right| \leq \sqrt{\frac{\prod_{j=1}^{s} [1 + 2\gamma_j \zeta(\alpha)] - 1}{r(\alpha, \boldsymbol{\gamma}; \mathbf{m})}} F_N(\mathcal{T}; \alpha, \boldsymbol{\gamma}; \xi). \tag{5}$$

Let now the equality $\lim_{N \to \infty} F_N(\mathcal{T}; \alpha, \boldsymbol{\gamma}; \xi) = 0$ holds for each even integer $\alpha \geq 2$ and for an arbitrary vector $\boldsymbol{\gamma}$ of positive weights. Hence, the inequality (5) gives that the equality $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi \mathbf{i}\langle \mathbf{m}, \mathbf{x}_n\rangle) = 0$ holds for each vector $\mathbf{m} \in \mathbf{Z}^s$ and $\mathbf{m} \neq \mathbf{0}$. Then from (3) the sequence ξ is uniformly distributed in $[0,1)^s$. Theorem 1 is completely proved.

Proof of Theorem 2. In Lemma 2 (ii) the Fourier coefficients of the function $G(\cdot)$ are presented. So, we have the equality

$$G(\alpha, \boldsymbol{\gamma}; \mathbf{x}) = \sum_{\mathbf{m}\in\mathbf{Z}^s} \widehat{G}(\alpha, \boldsymbol{\gamma}; \mathbf{m}) \exp(2\pi \mathbf{i}\langle \mathbf{m}, \mathbf{x}\rangle), \quad \forall \mathbf{x} \in [0,1)^s. \tag{6}$$

Let $\xi_N = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1}\}$ be an arbitrary net composed of $N \geq 1$ points in $[0,1)^s$. From (2) and (6) we consecutively obtain

$$\frac{1}{\prod_{j=1}^{s}[1+2\gamma_j\zeta(\alpha)]-1} \frac{1}{N^2} \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} G(\alpha, \boldsymbol{\gamma}; \mathbf{x}_n - \mathbf{x}_m)$$
$$= \frac{1}{\prod_{j=1}^{s}[1+2\gamma_j\zeta(\alpha)]-1} \frac{1}{N^2} \sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sum_{\mathbf{m}\in\mathbf{Z}^s} \widehat{G}(\alpha, \boldsymbol{\gamma}; \mathbf{m}) \exp(2\pi \mathbf{i}\langle \mathbf{m}, \mathbf{x}_n - \mathbf{x}_m\rangle)$$
$$= \frac{1}{\prod_{j=1}^{s}[1+2\gamma_j\zeta(\alpha)]-1} \sum_{\mathbf{m}\in\mathbf{Z}^s, \mathbf{m}\neq\mathbf{0}} r(\alpha, \boldsymbol{\gamma}; \mathbf{m}) \left| \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi \mathbf{i}\langle \mathbf{m}, \mathbf{x}_n\rangle) \right|^2$$
$$= F_N^2(\mathcal{T}; \alpha, \boldsymbol{\gamma}; \xi_N).$$

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Department of Informatics
Faculty of Science and Mathematics
St. St. Cyril and Methodius University
5, Arhimedova Str.
P.O. Box 162
1000 Skopje, Macedonia
e-mail: vassgrozdanov@yahoo.com

**Department of Mathematics*
Neofit Rilski South-West University
66, I. Mihailov Str.
2700 Blagoevgrad, Bulgaria
e-mail: vesna@ii.edu.mk