A NUMERICAL TEST ABOUT THE GENERALIZED B-ADIC DIAPHONY OF THE SEQUENCE OF HALTON

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ABSTRACT

In the present paper the authors give numerical results about the distribution of a special type of a sequence- the so-called sequence of Halton in comparison with theoretical estimations, obtained by theoretical examinations presented before. The so-called "generalized *b*-adic diaphony", which is based on the Walsh functional system over finite groups is taken as a measure for uniform distribution of sequences. The obtained numerical results show that the theoretical bounds about the generalized *b*-adic diaphony of the sequence of Halton give a frame in which numerical computations are settled.

I. WALSH FUNCTIONS OVER FINITE GROUPS

As a main tool of our work we will use the Walsh functional system over finite abelian groups. Following Larcher, Niederreiter and W. Ch. Schmid [1] and Larcher and Pirsic [2] we will recall the concept of this functional system. For a given integer $m \geq 1$, let $\{b_1,\ldots,b_m : 1 \leq l \leq m,\ b_l \geq 2\}$ be a set of fixed integers. For $1 \leq l \leq m$ let $Z_{b_l} = \{0,1,\ldots,b_l-1\}$, $\oplus_{b_l}$ be the operation summation mod b_l of the elements of the set Z_{b_l} and $(Z_{b_l}, \oplus_{b_l})$ is the discrete cyclic group of order b_l. Let $G = Z_{b_1} \times \ldots \times Z_{b_m}$ and for each pair $\boldsymbol{g}=(g_1,\ldots,g_m) \in G$ and $\boldsymbol{y}=(y_1,\ldots,y_m) \in G$ let us set $\boldsymbol{g} \oplus_G \boldsymbol{y} = (g_1 \oplus_{b_1} y_1, \ldots, g_m \oplus_{b_m} y_m)$. Then, $(G, \oplus_G)$ is a finite abelian group of order $b = b_1 b_2 \ldots b_m$. For the defined base b let us denote $\boldsymbol{Z}_b = \{0,1,\ldots,b-1\}$ and let $\varphi: \boldsymbol{Z}_b \to G$ be an arbitrary bijection with the condition $\varphi(0)=\mathbf{0}$. For $\boldsymbol{g},\boldsymbol{y} \in G$ let the character $\chi_{\mathrm{g}}(\mathrm{y})$ be defined by $\chi_{\mathrm{g}}(\mathrm{y}) = \prod_{l=1}^{m} \exp(2\pi I \frac{g_l y_l}{b_l})$. For an arbitrary real $x \in [0,1)$ we will use the b-adic representation $x = \sum_{i=0}^{\infty} x_i b^{-i-1}$, where for $i \geq 0$, $x_i \in \{0,1,\ldots,b-1\}$ and for infinitely many values of i, $x_i \neq b-1$. The b-adic representations of the reals in $[0,1)$ will be subordinated to this rule everywhere in our work.

Definition 1. For an arbitrary integer $k \geq 0$ and real $x \in [0,1)$ with the b-adic representations

$$k = \sum_{i=0}^{\upsilon} k_i b^i \text{ and } x = \sum_{i=0}^{\infty} x_i b^{-i-1},$$

where for $0 \leq i \leq \nu$, $k_i \in \{0,1,\ldots,b-1\}$, $k_\nu \neq 0$, the function ${}_{G,\varphi}wal_k(x) : [0,1) \to \mathbf{C}$ is defined in the following way

$${}_{G,\varphi}wal_k(x) = \prod_{i=0}^{\upsilon} \chi_{\varphi(k_i)}(\varphi(x_i)).$$

The set $W_{G,\varphi} = \{{}_{G,\varphi}wal_k : k=0,1,\ldots\}$ is called the Walsh functional system over the finite group G with respect to the bijection φ. Let $s \geq 1$ be a fixed integer and s will denote the dimension everywhere in the paper. Let $\boldsymbol{N_0}$ be the set of non-negative integers. The multivariate Walsh functions over the finite group G with respect to the bijection φ are defined by multiplication of the corresponding univariate functions, i.e. for a vector $\boldsymbol{k}=(k_1,\ldots,k_s) \in \boldsymbol{N_0}^s$ and $\boldsymbol{x}=(x_1,\ldots,x_s) \in [0,1)^s$ we set

$${}_{G,\varphi}wal_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{\mu=1}^{s} {}_{G,\varphi}wal_{k_\mu}(x_\mu).$$

For any dimension $s \geq 1$ the system $\{{}_{G,\varphi}wal_{\mathbf{k}} : \mathbf{k} \in \mathbf{N}_0^s\}$ is a complete orthonormal functional system on $L_2([0,1)^s)$. In the case when m=1, $G=\mathbf{Z}_b$ and $\varphi=\mathbf{id}$ the identity of the set $\mathbf{Z}_b$ in itself, the obtained system $W_{Z_b,\mathrm{id}}$ is the system W(b) of the Walsh functions in base b defined by Chrestenson [3]. The set W(2) is the original Walsh [4] functional system. So, for each integer $k \geq 0$ with the b-adic representation $k = \sum_{i=0}^{\upsilon} k_i b^i$ and $x = \sum_{i=0}^{\infty} x_i b^{-i-1}$, where for $0 \leq i \leq \nu$, $k_i \in \{0,1,\ldots,b-1\}$, $k_\nu \neq 0$, the Walsh function ${}_b wal_k : [0,1) \to \mathbf{C}$ is defined as ${}_b wal_k(x) = \prod_{i=0}^{\upsilon} \exp(2\pi I \frac{x_i k_i}{b})$.

II. INTRODUCTION

Following Kuipers and Niederreiter [5] we will recall the concept for uniformly distributed sequences. Let $\xi = (\boldsymbol{x}_i)_{i \geq 0}$ be an arbitrary sequence of points in $[0,1)^s$. For each integer $N \geq 1$ and an arbitrary subinterval J of $[0,1)^s$ with a volume $\mu(J)$, we denote by $A(\xi,J,N)$ the number of the points $\boldsymbol{x}_n$ of the sequence ξ whose indices n satisfy the inequalities $0 \leq n \leq N-1$ and belong to the interval J. The sequence ξ is called uniformly distributed in $[0,1)^s$ if the equality $\lim_{N \to \infty} \frac{A(\xi,J,N)}{N} = \mu(J)$ holds for every subinterval J of $[0,1)^s$. In general, the discrepancy and the diaphony are quantitative measures for uniform distribution of sequences in $[0,1)^s$. Zinterhof [6] uses the trigonometric functional system to introduce the "classical" diaphony.

The different kinds of the diaphony which is based on using orthonormal functional systems in base $b \geq 2$ as the dyadic diaphony introduced by Hellekalek and Leeb [7], the b-adic diaphony, see Grozdanov and Stoilova [8], the generalizations of the b-adic diaphony, see Grozdanov, Nikolova and Stoilova [9] have been considered so far as quantitative measures for uniform distribution of sequences in $[0,1)^s$. As a quantitative measure presenting the quality of the distribution of the points of a sequence in $[0,1)^s$ we will expose and use the so-called generalized b-adic diaphony. So, following [9] we recall the next definition.

Definition 2. For each integer $N \geq 1$ the generalized b-adic diaphony $F_N(W_{G,\varphi};\xi)$ of the first N elements of the sequence $\xi=(\boldsymbol{x}_i)_{i\geq 0}$ in $[0,1)^s$ is defined by

$$F_N(W_{G,\varphi};\xi) = \sqrt{\frac{1}{(b+1)^s-1} \sum_{\boldsymbol{k}\in \boldsymbol{N}_0^{\ s}, \boldsymbol{k}\neq \boldsymbol{0}} \rho(\boldsymbol{k}) \left| \frac{1}{N} \sum_{i=o}^{N-1} {}_{G,\varphi}wal_{\boldsymbol{k}}(\boldsymbol{x}_i) \right|^2}$$

where for a vector $\boldsymbol{k}=(k_1,\dots,k_s)\in \mathrm{N}_0^{s}$, $\rho(\boldsymbol{k})=\Pi^s_{\mu=1}\rho(k_\mu)$, and for an arbitrary integer $k\geq 0$

$$\rho(k) = \begin{cases} 1; k=0 \\ b^{-2g}; \forall k, b^g \leq k < b^{g+1}, g \in \boldsymbol{N}_0 \end{cases}$$

In [6] it is proved that the next theorem holds.

Theorem 1. The sequence ξ of points in $[0,1)^s$ is uniformly distributed in $[0,1)^s$ if and only if the next equality holds

$$\lim_{N\to\infty} F_N(W_{G,\varphi};\xi) = 0 .$$

Figure 1: Well distributed sequence of 16 points

III. Estimations for the generalized b-adic diaphony of the sequence of Halton

Definition 3. Let $b\geq 2$ be an arbitrary fixed integer. The sequence of Halton $H_b=H_b(i)_{i\geq 0}$ is defined as follows: if an arbitrary non-negative integer i has the b-adic representation $i=\sum_{j=0}^{\infty} a_j(i) b^j$ we replace $H_b(i)=\sum_{j=0}^{\infty} a_j(i) b^{-j-1}$.

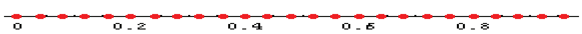

Figure 2: First 25 points from the Halton sequence, b=5

The sequence H_b was introduced by Halton.

According to the results exposed in [10] and [11], the following two theorems hold.

Theorem 2. For each integer $N\geq 1$ the generalized b-adic diaphony of the sequence of Halton H_b satisfies the inequality

$$F_N(W_{G,\varphi};H_b) \leq \tfrac{1}{N}\sqrt{c_1(b)Log[(b-1)N+1]+c_2(b)} ,$$

where the constants $c_1(b)$ and $c_2(b)$ are defined respectively as

$$c_1(b) = \frac{2b^2-2b-1}{Log(b)},\ c_2(b) = \frac{(b-1)^3}{b} .$$

Theorem 3. For infinitely many values of integer $N\geq 1$ the generalized b-adic diaphony of the sequence of Halton H_b satisfies the inequality

$$F_N(W_{G,\varphi};H_b) > \tfrac{1}{N}\sqrt{c_3(b)Log[(b^2-1)N+1]+c_4(b)} ,$$

where the constants $c_3(b)$ and $c_4(b)$ are defined respectively as

$$c_3(b) = \frac{(b^2-2)^2}{2(b+1)^3 b^2 (b-1) Log(b)} ,$$

$$c_4(b) = -\frac{(b^2-2)^2}{(b+1)^4 b^2 (b-1)^2} .$$

IV. Numerical tests for the the generalized b-adic diaphony of the sequence of Halton

In this paper we have examined some examples of the Halton sequence H_b (for different basis b=2,3,6,10,16) in the special case when the dimensionality of the points is one i.e. s=1; bijection φ is the identical bijection. In the numerical computations values of the generalized b-adic diaphony $F_N(W_{G,\varphi};H_b)$ of the first N elements of the H_b, are obtained when infinite sum in the definition of the generalized b-adic diaphony $F_N(W_{G,\varphi};H_b)$ will be replaced with a finite sum of 100 members. In the following text we will use the word "diaphony" with a meaning of "the generalized b-adic diaphony".

A. Results for case b=2

The first 16 elements of the Halton sequence, in the case when base b=2 are shown on the next figure, and on Fig.1.

$$\left\{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \frac{1}{16}, \frac{9}{16}, \frac{5}{16}, \frac{13}{16}, \frac{3}{16}, \frac{11}{16}, \frac{7}{16}, \frac{15}{16}\right\}$$

Figure 3: First 16 points from the Halton sequence, b=2

The values (with some precision) of the diaphony of the first N elements (N=100,200,…,1700) of the Halton sequence are shown on Fig.4.

N	Diaphony of first N el.
100.	0.0123269
200.	0.00567684
300.	0.0044145
400.	0.0022964
500.	0.00234854
600.	0.00198191
700.	0.001687
800.	0.00108253
900.	0.00108698
1000.	0.00113537
1100.	0.00118231
1200.	0.000871521
1300.	0.00096274
1400.	0.000682904
1500.	0.000838525
1600.	0.000441942
1700.	0.000739875

Figure 4: Table of diaphony of the first *N* elements of the Halton sequence, *b*=2

The following figure (Fig.5) shows upper (red line), lower (green line) bounds and values of the diaphony, presented on the previous figure.

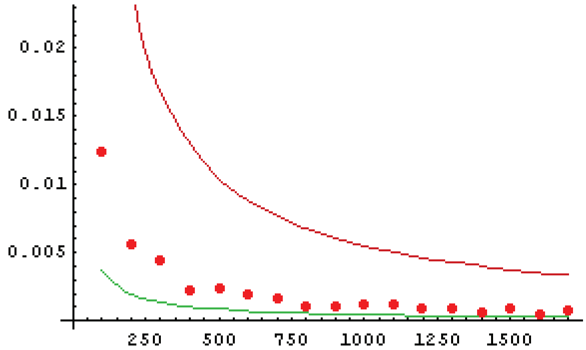

Figure 5: Diaphony of the first *N* elements of the Halton sequence, *b*=2

It is obvious that numerical results obtained in these computations, shown with red points on Fig.5, are settled exactly between two theoretical estimations - the lower and the upper bounds. According to this example, the lower bound is a better approximation than the upper bound. Results obtained from computations with 200 summands go with the previous conclusion.

B. *Results for case b=3*

The first 20 elements of the Halton sequence, in the case when base *b*=3 are shown on the next figure, and on Fig.7.

$$\left\{0, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{4}{9}, \frac{7}{9}, \frac{2}{9}, \frac{5}{9}, \frac{8}{9}, \frac{1}{27}, \frac{10}{27}, \frac{19}{27}, \frac{4}{27}, \frac{13}{27}, \frac{22}{27}, \frac{7}{27}, \frac{16}{27}, \frac{25}{27}, \frac{2}{27}, \frac{11}{27}\right\}$$

Figure 6: First 20 points from the Halton sequence, *b*=3

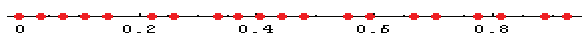

Figure 7: First 20 points from the Halton sequence, *b*=3

The values of the diaphony of the first *N* elements (*N*=100,200,…,1700) of the Halton sequence are shown on Fig.8.

N	*Diaphony of first N el.*
100.	0.0149381
200.	0.00797914
300.	0.00428082
400.	0.00366215
500.	0.00307501
600.	0.00237821
700.	0.00200537
800.	0.00179707
900.	0.00119696
1000.	0.00129245
1100.	0.00145309
1200.	0.00109398
1300.	0.00110131
1400.	0.00110013
1500.	0.000965597
1600.	0.0009685
1700.	0.000584894

Figure 8: Table of the diaphony of the first *N* elements of the Halton sequence, *b*=3

The following figure shows upper (red) and lower (green) bounds and values of the diaphony, presented on Fig.8.

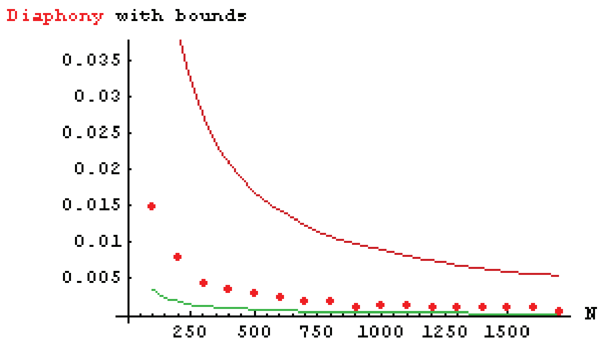

Figure 9: Diaphony of the first *N* elements of the Halton sequence, *b*=3

C. *Results for case b=6*

The first 20 elements of the Halton sequence, in the case when base *b*=6 are shown on Fig.10 and on Fig.11.

$$\left\{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \frac{1}{36}, \frac{7}{36}, \frac{13}{36}, \frac{19}{36}, \frac{25}{36}, \frac{31}{36}, \frac{1}{18}, \frac{2}{9}, \frac{7}{18}, \frac{5}{9}, \frac{13}{18}, \frac{8}{9}, \frac{1}{12}, \frac{1}{4}\right\}$$

Figure 10: First 20 points from the Halton sequence, *b*=6

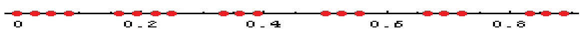

Figure 11: First 20 points from the Halton sequence, *b*=6

The values of the diaphony of the first *N* elements (*N*=100,200,…,1000) of the Halton sequence are shown on Fig.12.

N	*Diaphony of first N el.*
100.	0.0172971
200.	0.00852726
300.	0.00469765
400.	0.00369266
500.	0.00317178
600.	0.00221352
700.	0.00257218
800.	0.0021352
900.	0.0006415
1000.	0.00172971

Figure 12: Table of the diaphony of the first *N* elements of the Halton sequence, *b*=6

The following figure shows upper (red line), lower (green line) bounds and values of the diaphony, presented on the previous figure.

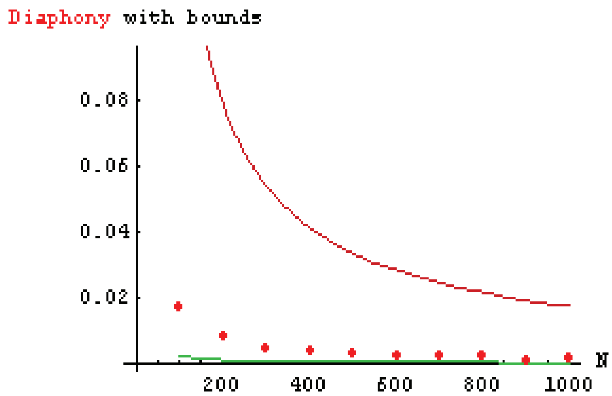

Figure 13: Diaphony of the first *N* elements of the Halton sequence, *b*=6

D. Results for case b=10

The first 20 elements of the Halton sequence, in the case when base *b*=10 are shown on the next figure, and on Fig.14.

$$\left\{0, \frac{1}{10}, \frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}, \frac{1}{100}, \frac{11}{100}, \frac{21}{100}, \frac{31}{100}, \frac{41}{100}, \frac{51}{100}, \frac{61}{100}, \frac{71}{100}, \frac{81}{100}, \frac{91}{100}\right\}$$

Figure 14: First 20 points from the Halton sequence, *b*=10

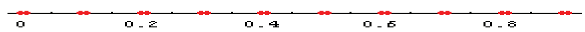

Figure 15: First 20 points from the Halton sequence, *b*=10

The values of the diaphony of the first *N* elements (*N*=100,200,…,1000) of the Halton sequence are shown on Fig.16.

N	*Diaphony of first N el.*
100.	0.00447214
200.	0.00394844
300.	0.00324416
400.	0.00255834
500.	0.00204667
600.	0.00170556
700.	0.00139036
800.	0.00098711
900.	0.000496904
1000.	0.

Figure 16: Diaphony of the first *N* elements of the Halton sequence, *b*=10

The following figure shows upper (red) and lower (green) bounds and values of the diaphony, for *b*=10, presented on Fig.16.

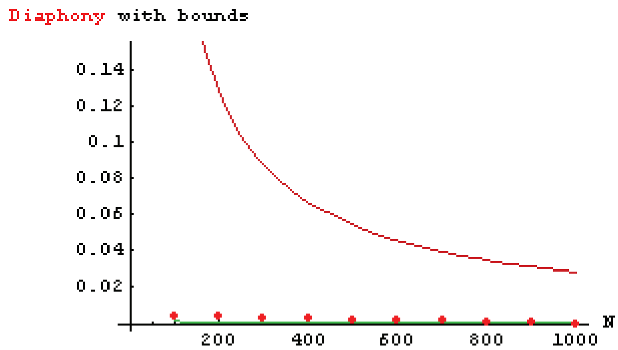

Figure 17: Diaphony of the first *N* elements of the Halton sequence, *b*=10

E. Results for case b=16

The first 20 elements of the Halton sequence, in the case when base *b*=16 are shown on the next figure, and on Fig.19.

$$\left\{0, \frac{1}{16}, \frac{1}{8}, \frac{3}{16}, \frac{1}{4}, \frac{5}{16}, \frac{3}{8}, \frac{7}{16}, \frac{1}{2}, \frac{9}{16}, \frac{5}{8}, \frac{11}{16}, \frac{3}{4}, \frac{13}{16}, \frac{7}{8}, \frac{15}{16}, \frac{1}{256}, \frac{17}{256}, \frac{33}{256}, \frac{49}{256}\right\}$$

Figure 18: First 20 points from the Halton sequence, *b*=16

Figure 19: First 20 points from the Halton sequence, b=16

The values of the diaphony of the first N elements (N=100,200,…,1000) of the Halton sequence are shown on Fig.20.

N	*Diaphony of first N el.*
100.	0.0892005
200.	0.0324505
300.	0.0248565
400.	0.0243091
500.	0.0106499
600.	0.0190268
700.	0.0138326
800.	0.0114343
900.	0.0153794
1000.	0.00878337

Figure 20: Diaphony of the first N elements of the Halton sequence, b=16

The following figure shows upper (red) and lower (green) bounds and values of the diaphony, for b=16, presented on Fig.20.

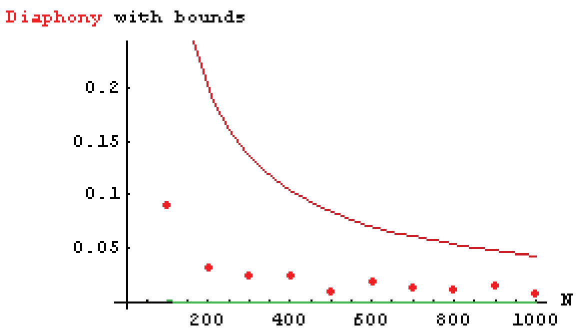

Figure 21: Diaphony of the first N elements of the Halton sequence, b=16

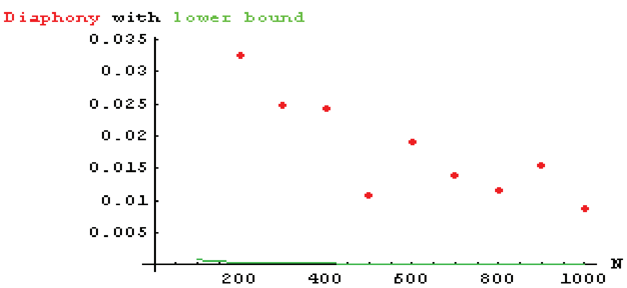

Figure 22: Diaphony of the first N elements of the Halton sequence, with lower bound, b=16

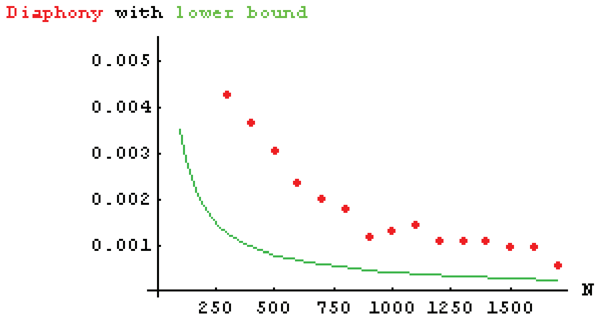

Figure 23: Diaphony of the first N elements of the Halton sequence, with lower bound, b=3

V. CONCLUSIONS

The results that were presented on the previous tables and figures obviously show that numerical results obtained in computations are settled exactly between the lower and upper theoretical estimations, given in [11]. According to all these examples, the lower bound is a better approximation than the upper bound.

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