VISUALIZATION OF A THEORETICAL RESULTS OBTAINED FOR A WEIGHTED DIAPHONY OVER CERTAIN SEQUENCES

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ABSTRACT

In this paper the authors give a visualization of a theoretical results obtained for a weighted diaphony over certain sequences: sequence of Van der Corput and net of Zaremba - Halton.

It is observed weighted diaphony as a numerical measure for uniform distribution of sequences with dimensions s=1 and s=2. The certain values of parameters and the certain choice of sequences are given.

The obtained theoretical results given by the examinations for the weighted diaphony‘s values are numerically tested and verified, with high level of accuracy.

Visualization of the theoretical results is made in the mathematical software *Mathematica.*

The high and low estimations of the weighted diaphony‘s values are verified.

I. INTRODUCTION

The discrepancy and the diaphony are quantitative measures for uniform distribution of sequences in $[0,1)^s$. Zinterhof [6] uses the trigonometric functional system

$\boldsymbol{\tau}=\{\exp(2\pi I(\mathbf{m},\mathbf{x})):\mathbf{m}=(m_1,m_2,...,m_s)\in\mathbf{Z}^s,\ \mathbf{x}=(x_1,x_2,...,x_s)\in[0,1)^s\}$

to introduce the "classical" diaphony, where $(\mathbf{m},\mathbf{x})$ is the inner product between the vectors $\mathbf{m}$ and $\mathbf{x}$.

The different kinds of the diaphony which is based on using orthonormal functional systems in base $b\geq 2$ as the dyadic diaphony introduced by Hellekalek and Leeb [7], the b-adic diaphony, see Grozdanov and Stoilova [8], the generalizations of the *b*-adic diaphony, see Grozdanov, Nikolova and Stoilova [9] have been considered so far as quantitative measures for uniform distribution of sequences in $[0,1)^s$.

As a quantitative measure presenting the quality of the distribution of the points of a sequence in $[0,1)^s$ we will observe and use the so-called weighted diaphony. So, following [12, 11] we will recall the next definition.

Definition 1. (The weighted diaphony) Let α be an arbitrary positive even integer, let $\boldsymbol{\gamma}$ be an arbitrary vector of positive weights $\boldsymbol{\gamma}=(\gamma_1,\ldots,\gamma_s)$ where $\gamma_1\geq\gamma_2\geq\ldots\geq\gamma_s>0$, and let $\boldsymbol{\tau}$ be the trigonometric functional system. For each integer $N\geq 1$ the weighted diaphony $F_N(\boldsymbol{\tau},\alpha,\boldsymbol{\gamma},\boldsymbol{\xi})$ of the first N elements of the sequence $\xi=(\mathbf{x}_i)_{i\geq 0}$ in $[0,1)^s$ is defined by

$$F_N(\tau,\alpha,\gamma,\xi)=\sqrt{\frac{1}{\prod_{j=1}^{s}(1+2\gamma_j\zeta(\alpha))-1}\sum_{\boldsymbol{k}\in Z^s,\boldsymbol{k}\neq\boldsymbol{0}}\rho(\alpha,\gamma,\boldsymbol{k})\left|\frac{1}{N}\sum_{i=o}^{N-1}\exp(2\pi I(\boldsymbol{k},\boldsymbol{x}_i))\right|^2}$$

where for a vector $\boldsymbol{k}=(k_1,\ldots,k_s)\in\boldsymbol{Z}^s$, $\rho(\alpha,\boldsymbol{\gamma},\boldsymbol{k})=\Pi^s_{j=1}\rho(\alpha,\gamma_j,k_j)$, and for an arbitrary $\gamma>0$ and an arbitrary integer k is defined by:

$$\rho(\alpha,\gamma,k)=\begin{cases}1;k=0\\ \frac{\gamma}{k^\alpha};k\neq 0\end{cases},$$

where $\zeta(\alpha)$ is well known Zeta function of Rieman.

Following Kuipers and Niederreiter [5] we will recall the concept for uniformly distributed sequences. Let $\xi=(\boldsymbol{x}_i)_{i\geq 0}$ be an arbitrary sequence of points in $[0,1)^s$. For each integer $N\geq 1$ and an arbitrary subinterval J of $[0,1)^s$ with a volume $\mu(J)$, we denote by $A(\xi,J,N)$ the number of the points $\boldsymbol{x}_n$ of the sequence ξ whose indices n satisfy the inequalities $0\leq n\leq N-1$ and belong to the interval J. The sequence ξ is called uniformly distributed in $[0,1)^s$ if the equality

$$\lim_{N\to\infty}\frac{A(\xi,J,N)}{N}=\mu(J)$$

holds for every subinterval J of $[0,1)^s$.

In [12] it is proved that the next theorem holds.

Theorem 1. The sequence $\boldsymbol{\xi}$ of points in $[0,1)^s$ is uniformly distributed in $[0,1)^s$ if and only if the next equality holds

$$\lim_{N\to\infty}F_N(\tau,\alpha,\gamma,\xi)=0.$$

Theorem 2. For an arbitary sequence $\boldsymbol{\xi}=(\boldsymbol{x}_i)_{i\geq 0}$ in $[0,1)^s$ the next two inequalities hold:

$$0\leq F_N(\boldsymbol{\tau},\alpha,\boldsymbol{\gamma},\boldsymbol{\xi})\leq 1.$$

A. *Some examples of “well” distributed sequence*

Ex.1.

Figure 1: “Well” distributed sequence of Halton of 25 points.

Ex.2.

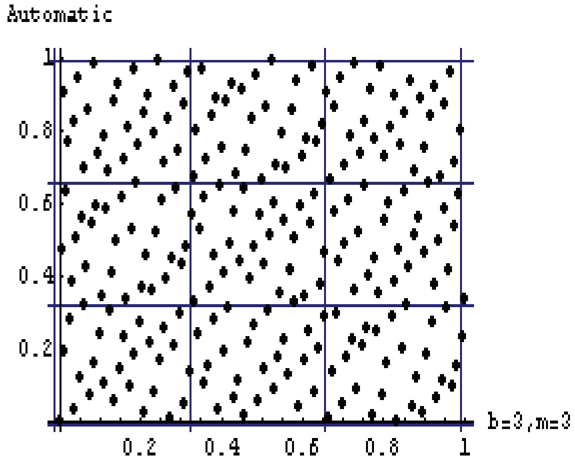

Figure 2: "Well" distributed (LPτ) net of 243 points, obtained for b=3 and m=3.

II. DEFINITIONS OF THE SEQUENCES S_B^{Σ} AND *ZH*

The next sequence was introduced by Faure in 1981.

Definition 2. The generalized b-adic sequence of **Van der Corput** $S_b^{\Sigma} = S_b^{\Sigma}(i)_{i\geq 0}$ is defined as follows: let $b\geq 2$ be an arbitrary fixed integer and let $\Sigma=(\sigma_i)_{i\geq 0}$ be a sequence of permutations of the set $\{0,1,\dots,b-1\}$. If an arbitrary non-negative integer i has the b-adic representation $i=\sum_{j=0}^{\infty} a_j(i)b^j$ we define

$$S_b^{\Sigma}(i)=\sum_{j=0}^{\infty}\sigma_j(a_j(i))b^{-j-1}\ .$$

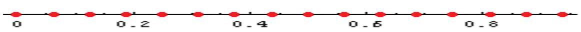

Figure 3: First 16 points from the Van der Corput sequence

In our experimentations we will observe the case when b=2 and $\Sigma= I$ is a sequence of identities, it is the case of the classical Van der Corput sequence S.

Definition 3. (The net of Zaremba- Halton) Let b>1 and ν>0 be an arbitrary integers, let $\alpha, \beta \in\{0,1,..,b-1\}$ be arbitrary and fixed. For $0\leq j\leq \nu-1$ we define $\mu_j= \alpha j+\beta(mod\ b)$ and put $\mu_b^{\alpha,\beta}=0.\ \mu_0\mu_{1\dots}\mu_{\nu-1}$. For an arbitrary integer i, $0\leq i\leq b^{\nu}-1$ with the b-adic representation $i=\sum_{j=0}^{\nu-1} a_j(i)b^j$ we put $\zeta_{b,\nu}(i)=\sum_{j=0}^{\nu-1} a_j(i)b^{-j-1}$ and we define $\zeta_{b,\nu}^{\alpha,\beta}(i)=\zeta_{b,\nu}(i)\oplus_b^{G,\varphi}\mu_b^{\alpha,\beta}$ and $\eta_{b,\nu}(i)=\dfrac{i}{b^{\nu}}$, where m>0, $\{b_1, b_2,\dots, b_m\}$ is a set of fixed integers, $G = Z_{b_1}\times Z_{b_2}\times\dots Z_{b_m}$ is a finite abelian group, φ is bijection and $\oplus_b^{G,\varphi}$ is the operation addition mod $b= b_1b_2\dots b_m$ in G.

For an arbitrary integer ν>0, arbitrary and fixed α, β $\in\{0,1,..,b-1\}$ we define the net

$$Z_{b,\nu}^{\alpha,\beta}=\{(\eta_{b,\nu}(i),\mu\zeta_{b,\nu}^{\alpha,\beta}(i)):0\leq i\leq b^{\nu}-1\}$$

which we will call the net of Zaremba-Halton over the group G with respect to the bijection φ and with the parameters α and β on base b.

In our numerical computations we considered only special cases of this net: ν=4, α=2, β=3, m=3, $b_1=2, b_2=3, b_3=2$ and this net we will noticed with ZH.

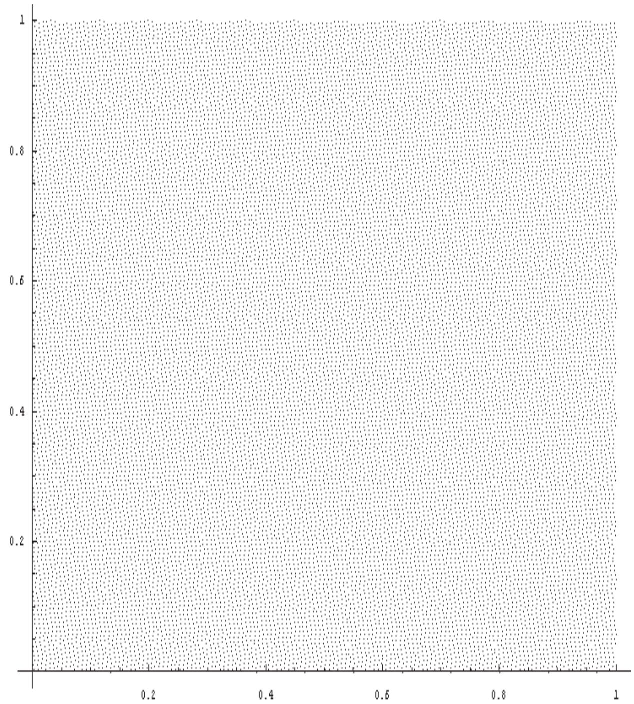

Figure 4: First 10000 points from net of Zaremba Halton

III. NUMERICAL TESTS FOR THE WEIGHTED DIAPHONY OF THE DEFINED SEQUENCES

In this paper we have examined some examples of the Van der Corput sequence S, and net of Zaremba Halton, for different number of the first N elements in the special case when the dimensionality of the points is s=1, or s=2 respectivelly. In the numerical computations values of the weighted diaphony $F_N(\boldsymbol{\tau},\alpha,\boldsymbol{\gamma},\xi)$ of the first N elements of these sequences, are obtained when infinite sum in the definition of the weighted diaphony $F_N(\boldsymbol{\tau},\alpha,\boldsymbol{\gamma},\xi)$ will be replaced with a finite sum of 1000 members. In the following text we will use the word "diaphony" with a meaning of "the weighted diaphony".

A. Results for Van der Corput sequence

The values (with some precision) of the diaphony of the first N elements (N=100,200,…,800) of the Van der Corput sequence are shown on Fig.5.

N	Tezhinska dijafonija
100	0.0001266538777886231756 6
200	0.0000341205684570496010 87
300	0.0000104061412324673638 48
400	$7.591885924728463 7 \times 10^{-6}$
500	$4.866399710868004131 0 \times 10^{-6}$
600	$3.258943727843826324 9 \times 10^{-6}$
700	$2.145315915908101620 7 \times 10^{-6}$
800	$1.815896941453097733 4 \times 10^{-6}$

Figure 5: Table of diaphony of the first N elements of the Van der Corput sequence, N<1000

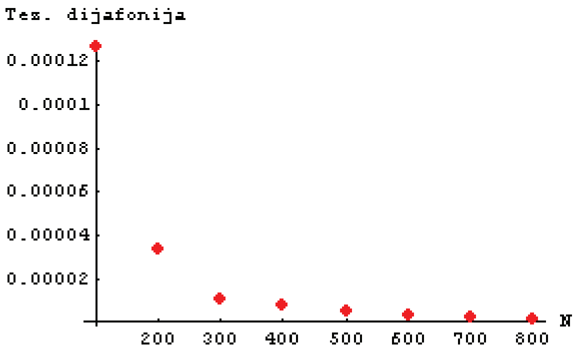

Figure 6: Diaphony of the first N elements of the Van der Corput sequence, N<1000

It is obvious that numerical results obtained in these computations, shown with red points on Fig.6, are settled exactly between two theoretical estimations - the lower '0" and the upper "1".

The values of the diaphony of the first N elements (N=1000,2000,3000,4000) of the Van der Corput sequence are shown on Fig. 7.

N	Tez. dijafonija
1000	$8.991330495520505422 4 \times 10^{-7}$
2000	$2.823461811112304445 7 \times 10^{-7}$
3000	$8.707030848341740725 9 \times 10^{-8}$
4000	8.54974×10^{-8}

Figure 7: Table of the diaphony of the first N elements of the Van der Corput sequence, 1000≤N≤4000

The following figure shows lower (green) bounds and values of the diaphony, presented on Fig. 7.

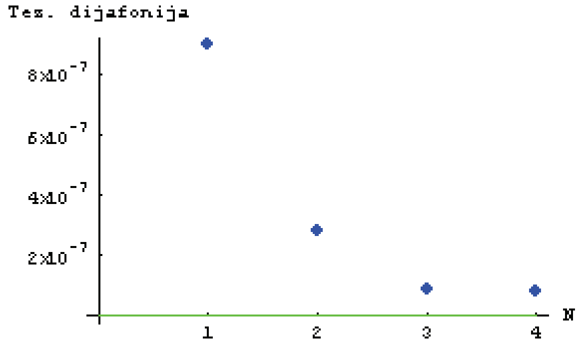

Figure 8: Diaphony of the first N elements of the Van der Corput sequence, 1000≤N≤4000

B. Results for the net of Zaremba- Halton

N	Dijafonija	
1000	8.9668459057682309000	$\times 10^{-7}$
2000	1.8766615251030611491	$\times 10^{-7}$
3000	8.2871915475714532811	$\times 10^{-8}$
4000	5.8344086581828877171	$\times 10^{-8}$
6000	3.0068181803973601242	$\times 10^{-8}$
8000	1.6268018842426959362	$\times 10^{-8}$
10000	6.8909768097704423332	$\times 10^{-9}$
12000	5.5296976167543190183	$\times 10^{-9}$

Figure 9: Table of the diaphony of the first N elements of the net of Zaremba- Halton

The following figure shows the values of the diaphony, of the net of Zaremba- Halton presented on Fig.9.

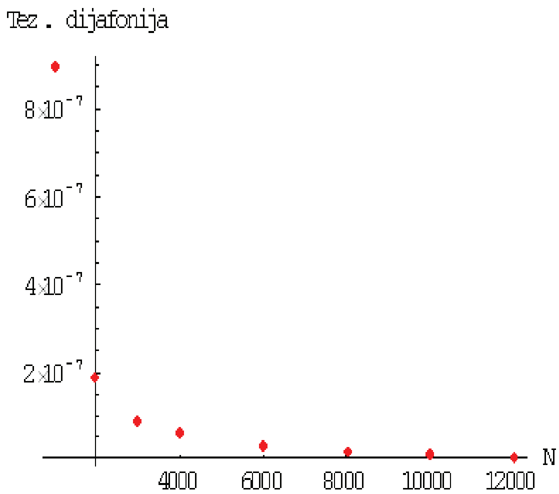

Figure 10: Diaphony of the first N elements of the net of Zaremba- Halton

IV. Conclusions

The results that were obtained on the previous tables and figures obviously show that numerical results presented in computations are settled exactly between the lower and upper theoretical estimations, given in Theorem 2. According to all these computations, the lower bound of the weighted diaphony is a better approximation than the upper bound.

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