Free Ternary Semicommutative Groupoids

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Abstract. *Free objects in the variety V of ternary groupoids defined by the identity $[xyz] \approx [zyx]$, are described. A proper characterization of the class of V-free objects by means of the class of V-injective groupoids is obtained as well.*

Key words: *ternary semicommutative groupoid, free/injective ternary semicommutative groupoid.*

1. INTRODUCTION AND PRELIMINARIES

An algebra with one ternary operation is called a ***ternary groupoid*** and is denoted by $(G,[\,])$. A ternary groupoid $(G,[\,])$ is ***semicommutative*** if $[xyz]=[zyx]$ for all $x,y,z \in G$ ([1]). The class of ternary semicommutative groupoids is a variety defined by the identity $[xyz] \approx [zyx]$ and it is denoted by V. We will give a description of free objects in this variety using the absolutely free ternary groupoid $\mathbf{F}_X = (F,[\,])$.

For the sake of completeness we will briefly present the construction of the absolutely free ternary groupoid $\mathbf{F}_X = (F,[\,])$ that is a special case of the construction of an absolutely free (n,m)-groupoid over a nonempty set X, that does not contain the symbol $[\,]$, for $n=3$ and $m=1$ ([4]). Let $F_0, F_1, \ldots, F_p, \ldots$ be a sequence of disjoint sets such that

$$F_0 = X \;,\; F_1 = \{(x_1,x_2,x_3) \mid x_1,x_2,x_3 \in F_0\}, \ldots,$$
$$F_{p+1} = \{(t_1,t_2,t_3) \mid t_1,t_2,t_3 \in F_0 \cup F_1 \cup \ldots \cup F_p \wedge (\exists i \in \{1,2,3\})\; t_i \in F_p\}.$$

If $t \in F_p$, then we say that t has a ***hierarchy*** p and denote it by $\chi(t)$.

Put $F = \bigcup \{F_p : p \geq 0\}$ and define a ternary operation $[\,]: F^3 \to F$ by:

$$t_1,t_2,t_3 \in F \;\Rightarrow\; [t_1,t_2,t_3] = (t_1,t_2,t_3).$$

Then $\mathbf{F}_X = (F, [\,])$ is a ternary groupoid. The set X is the generating set for $\mathbf{F}_X$ and $\mathbf{F}_X$ has the universal mapping property for the class of ternary groupoids over X. Therefore, $\mathbf{F}_X = (F, [\,])$ is free over X in the class of ternary groupoids. The groupoid $\mathbf{F}_X$ is *injective*, since the operation $[\,]$ is an injective mapping, i.e. $[a_1, a_2, a_3] = [b_1, b_2, b_3] \Rightarrow a_1 = b_1, a_2 = b_2, a_3 = b_3$. The set X is the set of prime elements in $\mathbf{F}_X$. An element $u \in F$ is said to be ***prime*** in $\mathbf{F}_X$ if and only if $u \neq [t_1, t_2, t_3]$ for all $t_1, t_2, t_3 \in F$. These two properties characterize all absolutely free ternary groupoids ([2], [3]).

Proposition 1.1. *A ternary groupoid* $(H, [\,])$ *is absolutely free if and only if* $(H, [\,])$ *is injective and the set of primes in* $(H, [\,])$ *is nonempty and generates* $(H, [\,])$.

We refer to this proposition as Bruck Theorem for the class of all ternary groupoids.

Define the ***length*** $|t|$, the *set* $P(t)$ *of* ***subterms*** and the set $var(t)$ of variables, for every $t \in F$, inductively in the following way:

$$t \in X \Rightarrow |t| = 1, \quad t = [t_1 t_2 t_3] \Rightarrow |t| = |t_1| + |t_2| + |t_3|,$$

$$t \in X \Rightarrow P(t) = \{t\}, \quad t = [t_1 t_2 t_3] \Rightarrow P(t) = \{t\} \cup P(t_1) \cup P(t_2) \cup P(t_3),$$

$$t \in X \Rightarrow var(t) = \{t\}, \quad t = [t_1 t_2 t_3] \Rightarrow var(t) = var(t_1) \cup var(t_2) \cup var(t_3).$$

2. A CONSTRUCTION OF CANONICAL OBJECTS IN V

A ternary semicommutative groupoid $\mathbf{R} = (R, [\,]^*)$ is said to be ***canonical*** if the following conditions are satisfied:

(C0) $X \subseteq R \subseteq F$ and $t \in R \Rightarrow P(t) \subseteq R$;

(C1) $[t_1 t_2 t_3] \in R \Rightarrow [t_1 t_2 t_3]^* = [t_1 t_2 t_3]$;

(C2) $\mathbf{R}$ is a V-free groupoid over X.

Define an ordering in F, denoted by $\leq$, in the following way. Let X be a linearly ordered set and $t, u \in F$. If $t, u \in X$, then $t \leq u$ in $F \Leftrightarrow t \leq u$ in X; if $\chi(t) < \chi(u)$, then $t \leq u$; if $\chi(t) = \chi(u) \geq 1$ and $t \neq u$, where $t = [t_1 t_2 t_3]$, $u = [u_1 u_2 u_3]$, then lexicographical ordering is used, with elements earlier in the lexicographical ordering coming first, i.e.

$$t < u \Leftrightarrow [t_1 < u_1 \vee (t_1 = u_1 \wedge t_2 < u_2)] \vee (t_1 = u_1 \wedge t_2 = u_2 \wedge t_3 < u_3)].$$

By induction on hierarchy one can show that F is a linearly ordered set.

Define a set R by:

$$R = X \cup \{ t \in F \setminus X \mid [t_1 t_2 t_3] \in P(t) \Rightarrow t_1 \leq t_3 \}.$$

Proposition 2.1. a) $X \subset R \subset F$; $t \in R \Rightarrow P(t) \subseteq R$.

b) $t_1, t_2, t_3 \in R \Rightarrow \left([t_1 t_2 t_3] \notin R \Leftrightarrow t_3 < t_1 \right)$.

c) $t_1, t_2, t_3 \in F \Rightarrow \left([t_1 t_2 t_3] \in R \Leftrightarrow t_1, t_2, t_3 \in R \wedge t_1 \leq t_3 \right)$.

Define a ternary operation $[\,]^*$ in R by:

$$t_1, t_2, t_3 \in R \Rightarrow [t_1 t_2 t_3]^* = \begin{cases} [t_1 t_2 t_3], & \text{if } t_1 \leq t_3 \\ [t_3 t_2 t_1], & \text{if } t_3 < t_1. \end{cases}$$

By a direct verification one can show that $(R, [\,]^*)$ is a ternary semicommutative groupoid, the set of prime elements in $(R, [\,]^*)$ coincides with X and generates $(R, [\,]^*)$. Also, $(R, [\,]^*)$ has the universal mapping property for V over X . Namely, if $\mathbf{G} = (G, [\,]') \in V$ and $\lambda : X \to G$ is a mapping, then there is a homomorphism $\psi : \mathbf{R} \to \mathbf{G}$ such that $\psi|_X = \lambda$. Let $\varphi : \mathbf{F}_X \to \mathbf{G}$ be the homomorphism that extends λ and let $\psi = \varphi|_R$. It suffices to show that $\varphi([t_1 t_2 t_3]^*) = [\varphi(t_1)\varphi(t_2)\varphi(t_3)]'$ for any $t_1, t_2, t_3 \in R$. If $[t_1 t_2 t_3] \in R$, then $[t_1 t_2 t_3]^* = [t_1 t_2 t_3]$, and so, $\varphi([t_1 t_2 t_3]^*) = \varphi([t_1 t_2 t_3]) =$ $= [\varphi(t_1)\varphi(t_2)\varphi(t_3)]'$. If $[t_1 t_2 t_3] \notin R$, then $[t_1 t_2 t_3]^* = [t_3 t_2 t_1]$, and so,

$$\varphi([t_1 t_2 t_3]^*) = \varphi([t_3 t_2 t_1]) = [\varphi(t_3)\varphi(t_2)\varphi(t_1)]' = [\mathbf{G} \in V] = [\varphi(t_1)\varphi(t_2)\varphi(t_3)]'.$$

Thus, we have proved the following

Theorem 2.1. *The ternary groupoid $(R, [\,]^*)$ is V-free over X and has a canonical form.*

3. A CHARACTERISATION OF FREE TERNARY SEMICOMMUTATIVE GROUPOIDS

We investigate some properties concerning the nonprime elements in $(R, [\,]^*)$ that are essential for introducing the notion of V-injectivity, and especially in which case the equality $[t_1 t_2 t_3]^* = [u_1 u_2 u_3]^*$ is fulfilled. By considering all the cases, one obtains:

Proposition 3.1. *Let t be a nonprime element in $(R,[\,]^*)$ and let (t_1,t_2,t_3) be a triple of divisors of t in $(R,[\,]^*)$. Then (u_1,u_2,u_3) is a triple of divisors of t if and only if $u_2=t_2$ and $\{u_1,u_3\}=\{t_1,t_3\}$.*

Therefore, $t=[t_1t_2t_3]\in R$ has one triple of divisors (t_1,t_2,t_3) if $t_1=t_3$ and

two triples of divisors (t_1,t_2,t_3) and (t_3,t_2,t_1) if $t_1\neq t_3$. The triple (t_1,t_2,t_3) of divisors of t in $(R,[\,]^*)$ coincides with the triple (t_1,t_2,t_3) in $(F,[\,])$. This is called the *canonical triple of divisors* in $(R,[\,]^*)$.

By Prop.3.1. the equalities $[t_1t_2u]^*=[t_1t_2v]^*$ and $[ut_2t_3]^*=[vt_2t_3]^*$ are equivalent with $\{t_1,u\}=\{t_1,v\}$ and $\{u,t_3\}=\{v,t_3\}$, respectively, which implies that $u=v$. Also, $[t_1ut_3]^*=[t_1vt_3]^*$ implies that $u=v$. Therefore:

Proposition 3.2. *The ternary groupoid $(R,[\,]^*)$ is left, right and middle cancellative.*

We use Prop.3.1. to introduce the notion of V-injective ternary groupoid.

Definition 3.1. A ternary groupoid $\mathbf{H}=(H,[\,])$ is said to be V *-injective* if and only if $\mathbf{H}\in V$ and $[a_1a_2a_3]=[b_1b_2b_3]\Rightarrow a_2=b_2\wedge\{a_1,a_3\}=\{b_1,b_3\}$, for all $a_i,b_i\in H$, $i=1,2,3$.

By Prop.3.1 it follows that the V-canonical ternary groupoid $(R,[\,]^*)$ is V-injective. Every V-free ternary groupoid over X is isomorphic with $(R,[\,]^*)$ and thus, the following proposition holds.

Proposition 3.3. *Every V-free ternary groupoid over X is V-injective.*

We will use the following lemma in the proof of Bruck Theorem for the variety V.

Lemma 3.1. *Let $\mathbf{H}=(H,[\,])$ be V-injective ternary groupoid such that the set P of primes in $\mathbf{H}$ is nonempty and generates $\mathbf{H}$. If $C_0=P$, $C_1=[C_0C_0C_0]$, …, $C_{k+1}=\{a\in H\setminus P:\ (d_1,d_2,d_3)$* is a triple of divisors of $a\Rightarrow$

$$\Rightarrow\{d_1,d_2,d_3\}\subset C_0\cup\ldots\cup C_k\wedge\{d_1,d_2,d_3\}\cap C_k\neq\varnothing\},$$

then $H=\bigcup\{C_k:k\geq 0\}$, where $C_k\neq\varnothing$ for every $k\geq 0$, and $C_i\cap C_j=\varnothing$ for $i\neq j$.

Proof. If $k=1$, then $a\in C_2$ if and only if $a=[d_1d_2d_3]$ for some $d_1,d_2,$ $d_3\in C_0\cup C_1$ and at least one $d_i\in C_1$. Then $[d_1d_2d_3]$ belongs to some of the sets $[C_0C_0C_1]$, $[C_0C_1C_0]$, $[C_1C_0C_0]$, $[C_0C_1C_1]$, $[C_1C_0C_1]$, $[C_1C_1C_0]$, $[C_1C_1C_1]$. Therefore, $a\in C_2$ if and only if a is in the union of these sets, i.e. C_2 equals to that union. Inductively, the set C_{k+1}, for any $k\geq 1$, is equal to the union of the sets $[C_pC_qC_r]$, where $p,q,r\in\{0,1,\ldots,k\}$ and at least one of the sets C_p, C_q, C_r is the set C_k. Hence, $C_i\cap C_{k+1}=\varnothing$. Therefore, $C_i\cap C_j=\varnothing$, for $i\neq j$. Since P generates $\mathbf{H}$, one can put $H=\bigcup\limits_{k\geq 0}P_k$, where $P_0=P$, $P_{k+1}=P_k\cup[P_kP_kP_k]$. By induction on k one can show that $P_k=C_0\cup C_1\cup\ldots\cup C_k$. Hence, $H=\bigcup\{C_k:k\geq 0\}$, where $C_k\neq\varnothing$, for any $k\geq 0$, and $C_i\cap C_j=\varnothing$ for $i\neq j$.

Theorem 3.1. (Bruck theorem for V**)** *A semicommutative ternary groupoid* $\mathbf{H}$ *is* V *-free if and only if* $\mathbf{H}$ *is* V *-injective and the set* P *of prime elements in* $\mathbf{H}$ *is nonempty and generates* $\mathbf{H}$*.*

Proof. The "if part" follows directly from Prop.3.3 and the fact that $\mathbf{H}\cong\mathbf{R}$. For the "only if part" we use Lemma 3.1.: $H=\bigcup\{C_k:k\geq 0\}$. It remains to show that $\mathbf{H}$ has the universal mapping property for V over P. Let $\mathbf{G}=(G,[\,]')$ be a semicommutative ternary groupoid and $\lambda:P\to G$ be a mapping. Define a sequence of mappings $\varphi_k:C_k\to G$ for $k\geq 0$, inductively by: $\varphi_0=\lambda$ and let $\varphi_i:C_i\to G$ be defined for every $i\leq k$. If $a\in C_{k+1}$ and (d_1,d_2,d_3) is a triple of divisors of a, where $d_1\in C_p$, $d_2\in C_q$, $d_3\in C_r$, then $p,q,r\leq k$. Putting $\varphi_{k+1}([d_1d_2d_3])=[\varphi_k(d_1)\varphi_k(d_2)\varphi_k(d_3)]'$, we obtain that $\varphi=\cup\{\varphi_i:i\geq 0\}$ is a mapping from H into G that is a homomorphism. If $a=[d_1d_2d_3]\in H$, then (d_1,d_2,d_3) is a triple of divisors of a in $\mathbf{H}$ and $\varphi(a)=\varphi([d_1d_2d_3])=[\varphi_k(d_1)\varphi_k(d_2)\varphi_k(d_3)]'=[\varphi(d_1)\varphi(d_2)\varphi(d_3)]'$. Since $\mathbf{H}$ is V-injective it follows that (d_3,d_2,d_1) is a triple of divisors of a as well and that $\varphi([d_3d_2d_1])=[\varphi(d_3)\varphi(d_2)\varphi(d_1)]'=[G\in\mathcal{K}_{sc}]=[\varphi(d_1)\varphi(d_2)\varphi(d_3)]'=$ $=\varphi([d_1d_2d_3])$. Thus, φ is a well defined mapping and a homomorphism. Hence, $\mathbf{H}$ is V-free over P.

The class of V -injective ternary groupoids is wider than the class of V -free groupoids, as the following example shows.

Let X be a countable set and let $(R, [\,]^*)$ be the V -canonical ternary groupoid over X. Define a set $H = \{t \in R : |P(t)| = 1\} \subset R$, a set $D \subset H^3$ by $D = \{(t_1, t_2, t_3) \in H^3 : P(t_i) \neq P(t_j) \text{ for a pair } i, j \in \{1,2,3\}\}$ and a relation θ in D by $(t_1, t_2, t_3)\, \theta\, (u_1, u_2, u_3) \Leftrightarrow t_2 = u_2 \wedge \{t_1, t_3\} = \{u_1, u_3\}$. It is easy to show that θ is an equivalence relation in D. The equivalence class $(t_1, t_2, t_3)^\theta = \{(t_1, t_2, t_3), (t_3, t_2, t_1)\}$, when $t_1 \neq t_3$ and $(t_1, t_2, t_3)^\theta = \{(t_1, t_2, t_3)\}$, when $t_1 = t_3$. The set of θ-equivalent classes is denoted by D^θ. It has the same cardinality as the set X and thus, there is an injection $\psi : D^\theta \to X$. Define an operation $[\,]'$ on H by: for $t_1, t_2, t_3 \in H$,

$$[t_1 t_2 t_3]' = \begin{cases} [t_1 t_2 t_3]^*, \text{ if } P(t_1) = P(t_2) = P(t_3) \\ \psi((t_1, t_2, t_3)^\theta), \text{ if } P(t_i) \neq P(t_j), \text{ for some pair } i, j \in \{1,2,3\}. \end{cases}$$

By a direct verification one can show that $\mathbf{H} = (H, [\,]')$ is a ternary groupoid that satisfies the conditions of Def.3.1. In the cases when the mapping ψ is an bijection, the set P of prime elements in $\mathbf{H}$ is empty. By Thm.3.1, $\mathbf{H} = (H, [\,]')$ is not V -free. Hence:

Proposition 3.4. *The class of V -free ternary groupoids is a proper subclass of the class of V -injective groupoids.*

This shows that Thm.3.1. gives a proper characterization of the class of free semicommutative ternary groupoids by means of the class of V -injective groupoids.

The obtained results can be easily generalized for semicommutative n -ary groupoids, i.e. n -ary groupoids defined by the identity $[x_1 x_2 ... x_{n-1} x_n] \approx [x_n x_2 ... x_{n-1} x_1]$.

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