ON A VARIETY OF GROUPOIDS OF RANK 1

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Abstract: A canonical description of free objects in the variety V of groupoids defined by the identity $xx^2 = x^2x^2$ is given. Injective groupoids in V and subgroupoids of free groupoids in V are considered. It is shown that the class of free groupoids in V is a proper subclass of the class of injective groupoids in V and that both classes are hereditary.

0. Introduction

Throughout the paper we denote by $\boldsymbol{F} = (F, \cdot)$ a given absolutely free groupoid with a basis B (i.e. groupoid free in the class of all groupoids). It is well-known ([1; L.1.5.]) that the following two conditions caracterizes $\boldsymbol{F}$: a) $\boldsymbol{F}$ *is injective* [1]; b) *the set* B *of prime elements in* $\boldsymbol{F}$ *is nonempty and generates* $\boldsymbol{F}$.

The subject of this paper is the variety of groupoids of rank 1,[2] defined by the identity [3]

$$xx^2 = x^2x^2, \tag{0.1}$$

which we denote by V . The paper is divided into three sections.

In Section 1 we give a description of V -free groupoids and show that they may be different.

In Section 2, the notion of V -injective groupoid (i.e. groupoid injective in V) is introduced. It is shown that: the class of V -injective groupoids is hereditary, every V -injective groupoid is infinite and the class of V -free groupoids is a proper subclass of the class of injective groupoids.

In Section 3 are considered subgroupoids of V -free (i.e. free in V) groupoids. We prove that: Bruck Theorem for V holds, the class of V -free groupoids is hereditary and that every V -free groupoid contains a subgroupoid with an infinite basis.

1. Free objects in V

For a construction of a free object in a variety V of groupoids with an axiom $f = g$, the following "procedure" is often convenient. We consider one of the two parts of the axiom of V as "more suitable", and, as a candidate for the carrier of the desired free object, we choose the set R of all elements $t \in F$ [4] which contain no parts with a form of the "unsuitable side of the axiom".

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[1] We do not define here the notions as: injective groupoid, prime element, length $|v|$ and set $P(v)$ of parts of $v \in F$, ... (see, for example [2]).

[2] A variety defined by identities wich contain only one variable is called a *variety of rank* 1.

[3] Here we use the usual abbreviations: $xx^2 = x(xx)$, $x^2x^2 = (xx)(xx)$.

[4] $F = (F, \cdot)$, as above, is an absolutely free groupoid with the free basis B.

Here we consider the variety V with an axiom $xx^2 = x^2x^2$. Choosing the left side as "unsuitable", we obtain:

$$R = \{\, t \in F\colon (\forall \alpha \in F)\ \alpha\alpha^2 \notin P(t)\,\}.^{5)} \tag{1.1}$$

By (1.1) immediately we obtain:

a) $(\forall\, t,u \in F)\{\, tu \in R \Leftrightarrow t,u \in R \ \&\ u \neq t^2 \,\}$.

b) $t,u \in R \Rightarrow \{\, tu \notin R \Leftrightarrow u = t^2 \,\}$.

c) $t \in R \Rightarrow (\forall k \in N)\ t^k \in R\}$ [6] (t^n is defined in $\boldsymbol{F}$ by: $t^1 = t,\ \ t^{n+1} = t^n t$).

Define an operation $*$ on R by:

$$t,u \in R \Rightarrow\ t * u = \begin{cases} tu, & \text{if } tu \in R \\ (t^2)^2, & \text{if } u = t^2 \end{cases} \tag{1.2}$$

By a direct verification we obtain that $\boldsymbol{R} = (R, *)$ is a groupoid, the equality (0.1) holds in $\boldsymbol{R}$ (i.e. $t*(t*t) = (t*t)*(t*t)$ is an identity in $\boldsymbol{R}$) which means that $\boldsymbol{R} \in$ V and for any mapping $\lambda : B \to G$ [7], there is a homomorphism $\varphi\colon \boldsymbol{F} \to \boldsymbol{G}$ which extends λ. Therefore:

Theorem 1. *$\boldsymbol{R} = (R, *)$ is a free groupoid in* V *with the basis B, and B coincides with the set of primes in* $\boldsymbol{R}$.

Bellow we will state some properties of the groupoid $\boldsymbol{R}$, but first we will consider the groupoid power $x^{(k)},\ k \geq 0$, defined by:

$$x^{(0)} = x,\quad x^{(k+1)} = x^{(k)}x^{(k)} = (x^{(k)})^2 \tag{1.3}$$

By induction on m and n one can show that, in any groupoid $\boldsymbol{G} = (G, \cdot)$, the following statement is true:

$$(\forall x \in G,\ m, n \geq 0)\quad (x^{(m)})^{(n)} = x^{(m+n)} \tag{1.4}$$

If $\boldsymbol{G} \in$ V, then by (1.3) and (0.1) one obtains:

$$(\forall x \in G,\ p \geq 0)\quad x^{(p)}x^{(p+1)} = x^{(p+2)}. \tag{1.5}$$

We say that an element $a \in G$ is a *power* in $\boldsymbol{G}$ if there are $b \in G$ and $k \geq 1$, such that $a = b^{(k)}$. If $b \in G$ is not a power in $\boldsymbol{G}$, i.e.

$$(\forall c \in G)\ (c = b^{(p)} \Rightarrow p = 0),$$

then we say that b is a *base* in $\boldsymbol{G}$.

As a special case of c), one obtains:

c') $t \in R, k \geq 1 \Rightarrow t^{(k)} \in R$.

Note that, if $t \in R$, t^n is the n-th power of t in $\boldsymbol{F}$; in this sence, $t_*^{\,n}$ is the n-th power of t in $\boldsymbol{R}$, defined by: $t_*^{\,1} = t,\ t_*^{\,n+1} = t_*^{\,n} * t$.

From (1.2) and c'), by induction on k, we obtain:

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[5] $P(t)$ is the set of parts (i.e. subterms) of t.

[6] N is the set of positive integers.

[7] The carrier of a given groupoid $\boldsymbol{S}$ is denoted by the same (light) letter S.

d) $(\forall t\in R,\ k\geq 0)\quad t_*^{(k)}=t^{(k)}$ and, for $k\geq 1$: $t_*^k=t^k$.

As a concequence of d), we obtain the following two statements.

Proposition 1.1. $(\forall u\in R)(\exists!(t,p)\in R\times N_0)$[8] $u=t_*^{(p)}$, *where* t *is a base in* $\boldsymbol{R}$.

Proposition 1.2. a) *If* $x\in R\setminus B$ *(i.e.* x *is not prime in* $\boldsymbol{R}$*) and if* x *is not a power, then there exists a unique pair* $(u,v)\in R^2$, *such that* $x=u*v$. *(In this case* $x=uv$.*)*

(We say that (u,v) is *the pair of divisors of* x and write $(u,v)\mid x$.)

b) *If* $x\in R$ *is a power,* $x=t^{(p+1)}$, $p\geq 0$, *then* $x=t^{(p)}*t^{(p)}$, *and* $(t^{(p)},\ t^{(p)})$ *is the pair of divisors of x.*

The class of groupoids free in V will be denoted by V_{free}. We will show the following

Propopsition 1.3. *If* $\boldsymbol{H}\in\mathrm{V}_{free}$ *with the basis* B, *then there exists a mapping* $x\mapsto|x|$ *from* H *into* N *such that*

$$|b|=1,\ |cd|\geq|c|+|d|, \tag{1.6}$$

for any $b\in B,\ c,d\in H$.

Proof. Let $\boldsymbol{R}=(R,*)$ be the V -canonical groupoid with the basis B (constructed above, Th.1) and let $x\mapsto|x|$ be the restriction of the mapping $|\ |:F\to N$. Let $t,\ u\in R$. Since $t,\ u\in F$, it follows that $|tu|=|t|+|u|$ and $|tt^2|=3|t|$. From (1.2) we obtain:

$$tu\in R\Rightarrow|t*u|=|t|+|u|,$$

$$tu\notin R\Rightarrow u=t^2\Rightarrow\left|t*t^2\right|=\left|(t^2)^2\right|=4\,|t|>3\,|t|.$$

This shows that $|t*u|\geq|t|+|u|$

Now, let $\boldsymbol{H}\in\mathrm{V}_{free}$ with the basis B. Since $\boldsymbol{H}$ is isomorphic with $\boldsymbol{R}$, it follows that (1.6) holds.

Remark 1.4. When one decides which side of the identity $xx^2=x^2x^2$ to consider as "suitable" one, it is natural to choose the "shorter" side, i.e. xx^2, and expect a shorter construction of V -free groupoid. However, it turns out the opposite, the construction is longer and more complicated.

Namely, let the first candidate for the carrier of V -free groupoid be the set F_1, defined by:

$$F_1=\{\,t\in F:(\forall\alpha\in F)\ (\alpha^2)^2\notin P(t)\,\}.$$

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[8] N_0 is the set of nonnegative integers.

If we define an operation $*_1$ on F_1 by:

$$t,u\in F_1 \Rightarrow \ t *_1 u = \begin{cases} tu, \text{ if } tu\in F_1 \\ \alpha\alpha^2, \text{ if } t=u=\alpha^2 \end{cases}$$

then we obtain that $\boldsymbol{F_1} = (F_1, *_1)$ is a groupoid. However, the equality (0.1), which has the form here

$$t *_1 (t *_1 t) = (t *_1 t) *_1 (t *_1 t) \tag{1.7}$$

is not satisfied in $\boldsymbol{F_1}$, i.e. $\boldsymbol{F_1} \notin \mathrm{V}$. Namely, if $t=\alpha^2$, the left side of (1.7) is $\alpha^2(\alpha\alpha^2)$ and the right one is $(\alpha\alpha^2)^2$. This result implies that

$$\alpha^2(\alpha\alpha^2) = (\alpha\alpha^2)^2 \tag{1.7'}$$

is an identity in V . This sugests a definition of a new "candidate" $\boldsymbol{F_2} = (F_2, *_2)$:

$$F_2 = \{\, t\in F_1 : (\forall\alpha\in F_1)\ (\alpha\alpha^2)^2 \notin P(t)\,\},$$

$$t,u\in F_2 \Rightarrow \ t *_2 u = \begin{cases} t *_1 u, \text{ if } t *_1 u\in F_2 \\ \alpha^2(\alpha\alpha^2), \text{ if } t=u=\alpha\alpha^2 \end{cases}$$

Checking (1.7) (when $*_1$ is substituted by $*_2$), we obtain $\boldsymbol{F_2} \notin \mathrm{V}$ and one more identity in V :

$$(\alpha\alpha^2)(\alpha^2(\alpha\alpha^2)) = ((\alpha^2(\alpha\alpha^2))^2 . \tag{1.7''}$$

Continuing this procedure, we can see a regularity in the concequences of the identity (1.7), which sugests to introduce a special kind of groupoid power $x \mapsto x^{<n>}$, defined by:

$$x^{<0>} = x, \quad x^{<1>} = x^2, \quad x^{<k+2>} = x^{<k>}x^{<k+1>} . \tag{1.8}$$

Using this, we defne the following infinite set of groupoids:

$$\{\boldsymbol{F_n} = (F_n, *_n) : n \geq 1\},$$

$$F_1 = \{\, t\in F : (\forall\alpha\in F)\ (\alpha^{<1>})^2 \notin P(t)\,\},$$

$$t,u\in F_1 \Rightarrow \ t *_1 u = \begin{cases} tu, \text{ if } tu\in F_1 \\ \alpha\alpha^2, \text{ if } t=u=\alpha^{<1>} \end{cases}$$

$$F_{n+1} = \{\, t\in F_n : (\forall\alpha\in F_n)\ (\alpha^{<n>})^2 \notin P(t)\,\},$$

$$t,u\in F_{n+1} \Rightarrow \ t *_{n+1} u = \begin{cases} t *_n u, \text{ if } tu\in F_{n+1} \\ \alpha^{<n+2>}, \text{ if } t=u=\alpha^{<n+1>}. \end{cases}$$

One can show that $\boldsymbol{F_{n+1}}$ is a groupoid such that $\boldsymbol{F_{n+1}} \notin \mathrm{V}$.

Using the fact that $F \supseteq F_1 \supseteq \ldots \supseteq F_n \supseteq \ldots$ and that $\boldsymbol{F_{n+1}}$ is "better" than $\boldsymbol{F_n}$, we obtain the following definition of the carrier R' of a free groupoid in V :

$$R' = \{\, t\in F : (\forall\alpha\in F, n\geq 1)\ (\alpha^{<n>})^2 \notin P(t)\,\} \ \left(= \bigcap \{F_n : n\geq 1\}\right).$$

(Note that it is not necessary to define the whole sequence, because the desired "good candidate" can be noticed after several steps.)

We define an operation $*$ on R' by

$$t,u\in R' \Rightarrow \ t*u=\begin{cases} tu, \text{ if } tu\in R' \\ \alpha^{<n+1>}, \text{ if } t=u=\alpha^{<n>},\ n\geq 1 \end{cases}$$

and obtain:

Proposition 1.5. *R' is a V -free groupoid with the basis B.*

Therefore, there exist at least two distinct V -free groupoids, R and R'. Since R and R' have the same basis B, they are isomorphic.

2. Injective groupoids in V

We obtain the class of injective groupoids in V from the class of V -free groupoids in the same way as in the variety of all groupoids (and, namely, by omiting the condition that the set of primes is a generating set).

Using Pr.1.1 – 1.2, we come to the following definition of injective groupoids in V.

We say that a groupoid $H=(H,\cdot)\in$ V is *injective in* V (i.e. V –injective) iff the following conditions are satisfied:

(i_1) $(\forall a\in H)(\exists!(b,p)\in H\times N_0)\ a=b^{(p)}$ and b is a base in H.

(We say that b is the *base* and p the *exponent* of a.)

(i_2) $b^{(p+2)}=cd$ iff $[c=d=b^{(p)}$ or $(c=b^{(p)}\ \&\ d=b^{(p+1)})]$.

(ii) If b is a base in H, then: $b^{(1)}=cd \Leftrightarrow c=d=b$.

(iii) If $a\in H$ is a base which is not prime in H, then

$$(\exists!(c,d)\in H^2) \quad a=cd,\ c\neq d\ .$$

The pair of divisors (c,d) of an element $a\in H$ which is not prime in H (we write: $(c,d)\big|\,a$) is defined as follows.

1) If b is a base in H and $p\geq 0$, then $(b^{(p)},b^{(p)})\Big|\,b^{(p+1)}$.

2) If $a=cd$ is a base in H, then $(c,d)\big|\,a$.

We denote by V_{inj} the class of injective groupoids. By Pr.1.1–1.2 we obtain:

Proposition 2.1. $V_{free}\subseteq V_{inj}$.

Proposition 2.2. *Let $H\in V_{inj}$, $Q\leq H$, $a=b^{(p)}\in Q$ and $b\notin Q$, where b is the base of a in H. If $r=\min\{k: b^{(k)}\in Q\}$, then $b^{(r)}$ is a prime in Q.*

As a corollary of Pr.2.2, we obtain the following

Proposition 2.3. *The class V_{inj} is hereditary.*

If $c^{(p)}=d^{(q)}$ and c, d are bases in $H\in V_{inj}$, then $c=d\ \&\ p=q$. Therefore, if a is a base in H, then the powers $a^{(n)}$, $n\geq 1$, are mutually distinct and thus the set $\{a,a^{(1)},a^{(2)},...\}$ is infinite. Therefore:

Proposition 2.4. *Every $\boldsymbol{H} \in \mathrm{V}_{inj}$ is infinite.*

Bellow we will give a *construction of* V *-injective groupoids* (and show that there are V -injective groupoids which are not V –free).

Let A be an infinite set and $H = A \times N_0$. Define a partial operation • on H by:

$$(a,p) \bullet (a,p) = (a, p+1),$$
$$(a,p) \bullet (a, p+1) = (a, p+2)$$

and put

$$D = \{((a,p),(b,q)) : a \neq b \ \ or \ \ (a = b \ \& \ q \notin \{p, \ p+1\}\}.$$

Since A and D have the same cardinality, there is an injection $\varphi: D \to A$ and we can put

$$(\forall (a,p),(b,q) \in D) \ (a,p) \bullet (b,q) = (\varphi((a,p),(b,q)), 0).$$

Then we obtain that $(H, \bullet)$ is a groupoid injective in V .

If φ is a bijection, then the set $A \times \{0\} \setminus \mathrm{im}\varphi$ of primes in H is empty, and thus $(H, \bullet)$ is not free in V .

This and Cor.2.1 proves the following

Theorem 2. *The class V_{free} is a proper subclass of V_{inj} .*

3. Subgroupoids of V –free groupoids

In this section we will show that the class V_{free} is hereditary, but first we will give a caracterizatin of V -free groupoids (analogous to a), b) in Introduction, for absolutely free groupoids).

Theorem 3 (Bruck Theorem for V). *A groupoid $\boldsymbol{H} \in \mathrm{V}$ is V -free iff:*

(i) ***H*** *is* V *–injective.*

(ii) *The set B of primes in* ***H*** *is nonempty and generates* ***H***.

Proof. If ***H*** is V -free with the basis B, then $\boldsymbol{H} \in \mathrm{V}_{inj}$ (by Th.2), B is the set of primes in ***H*** and generates ***H***.

For the converse, define an infinite sequence of subsets $B_1, B_2, \ldots$ of H:

$$B_1 = B, \ \ B_2 = \{cd : c, d \in B_1\},$$
$$B_{k+1} = \{a \in HH : (c,d) | a \Rightarrow \{c,d\} \subseteq B_1 \cup \ldots \cup B_k \ \& \ \{c,d\} \cap B_k \neq \varnothing\}.$$

Then the following statements are true:

1) $(\forall k \geq 1) \ B_k \neq \varnothing$; 2) $p \neq q \Rightarrow B_p \cap B_q = \varnothing$; 3) $H = \bigcup \{B_k : k \geq 1\}$.

Let $G \in \mathrm{V}$ and $\lambda : B \to G$ be a mapping. Defining a sequence of mappings $\varphi_k: B_k \to G$ inductively ($\varphi_1 = \lambda$; ...) and continuing in a similar way as in [2; Th.1], one can show that the mapping $\varphi = \bigcup \{\varphi_k : k \geq 1\}$ is a homomorphism from ***H*** into ***G*** which extends λ.

Below we assume that $\boldsymbol{H} \in \mathrm{V}_{free}$ and ***Q*** is a subgroupoid of ***H***.

Proposition 3.1. *The set P of primes in* ***Q*** *is nonempty and generates* ***Q***.

Proof. By Pr.1.3, there is a mapping in H with the property (1.6). Let $c \in \boldsymbol{Q}$ be such that

$$|c| = \{\min |a| : a \in Q\} \tag{3.1}$$

By (1.6) and (3.1), c is prime in $\boldsymbol{Q}$. Thus the set P of primes in $\boldsymbol{Q}$ is nonempty.

Denote by $\boldsymbol{T}$ the groupoid generated by P. Clearly, $T \subseteq Q$. We will show that $Q \subseteq T$, using induction on length.

First of all, $P \subseteq T$. If $b \in \boldsymbol{Q}$ and $|b| = 1$, then b is prime in $\boldsymbol{H}$, and thus b is prime in $\boldsymbol{Q}$, i.e. $b \in P$. Therefore $b \in T$. Suppose $c \in Q \ \& \ |c| \le k \Rightarrow c \in T$.

Let $c \in Q$ and $|c| = k+1$. If $c \in P$, then $c \in T$. Therefore let $c \in Q \setminus P$. Then $c = de \ \ (d, e \in Q)$, and:

$$|c| = |de| \ge |d| + |e| \Rightarrow |d|, |e| \le k \Rightarrow d, e \in T \Rightarrow c = de \in T.$$

Thus $Q \subseteq T$ and therefore P generates Q.

Proposition 3.2. $\boldsymbol{Q} \in \mathrm{V}_{free}$.

Proof. By Pr.2.1, $\boldsymbol{H} \in \mathrm{V}_{free}$ implies $\boldsymbol{H} \in \mathrm{V}_{inj}$, and by Pr.2.3, $\boldsymbol{Q} \in \mathrm{V}_{inj}$. Now, applying Bruck Theorem for V we obtain that $\boldsymbol{Q} \in \mathrm{V}_{free}$.

As a corollary of Pr.3.2 we obtain

Proposition 3.3. *The class* V_{free} *is hereditary.*

Finaly, we will prove the following

Proposition 3.4. *If* $\boldsymbol{H} \in \mathrm{V}_{free}$, *then there is a subgroupoid* Q *such that* Q *has an infinite basis.*

Proof. Let B be the basis of $\boldsymbol{H}$ and $a \in B$. Put

$$c_1 = a \cdot a, \ c_2 = c_1 a, \ldots, \ c_{k+1} = c_k a, \ldots$$

and let $\boldsymbol{Q}$ be the subgroupoid of $\boldsymbol{H}$ generated by the set $C = \{c_k : k \ge 0\}$. Then: 1) C is infinite; 2) all the elements in C are prime in $\boldsymbol{Q}$; 3) C is a basis of $\boldsymbol{Q}$.

Namely, the elements of C are mutually distinct $(c_p = c_q \Rightarrow p = q)$ and thus C is infinite. Secondly, every element $c_k \in C$ is a base in $\boldsymbol{Q}$: $c_1 = aa = a^2$, but $a \notin Q$ and thus c_1 is a base in $\boldsymbol{Q}$; $c_1 = a^2 a = a^3$, $a^2 \in Q$, but $a \notin Q$, and thus c_2 is a base in $\boldsymbol{Q}$; inductively, $c_{k+1} = a^k a$ is a base in $\boldsymbol{Q}$. Thirdly, C is the set of primes in $\boldsymbol{Q}$, $C \ne \varnothing$ and generates $\boldsymbol{Q}$. Therefore, C is the basis of $\boldsymbol{Q}$.

References

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