N. Shekutkovski, M. Shoptrajanov, Intrinsic Shape Property of Global Attractors in Metrizable Spaces, *Rus. J. Nonlin. Dyn.*, 2020, Volume 16, Number 1, 181–194

DOI: https://doi.org/10.20537/nd200114

Russian Journal of Nonlinear Dynamics, 2020, vol. 16, no. 1, pp. 181–194.
Full-texts are available at http://nd.ics.org.ru
DOI: 10.20537/nd200114

MATHEMATICAL PROBLEMS OF NONLINEARITY

MSC 2010: 54H20, 54C56, 37B20, 37B25

Intrinsic Shape Property of Global Attractors in Metrizable Spaces

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This paper concerns the connection between shape theory and attractors for semidynamical systems in metric spaces. We show that intrinsic shape theory from [6] is a convenient framework to study the global properties which the attractor inherits from the phase space. Namely, following [6] we'll improve some of the previous results about the shape of global attractors in arbitrary metrizable spaces by using the intrinsic approach to shape which combines continuity up to a covering and the corresponding homotopies of first order.

Keywords: intrinsic shape, regular covering, continuity over a covering, attractor, proximate net

1. Introduction

The pioneering work in the study of dynamical systems of Poincaré, later continued by Morse, Smale and Conley, can be considered as landmarks in the study of dynamical systems through their phase portraits, which are objects of a topological nature. This approach gave rise to a whole new branch, where tools like homotopy theory, homology, and later on shape theory, played a prominent role in the investigations of dynamical systems.

The use of shape theory in the study of dynamical systems was initiated by Hastings in [7]. Other authors showed how to apply shape theory to obtain global properties of attractors in Refs. [9–11]. Shape theory was related to differential equations in [12] and it is the main tool used in [13, 14] to define a Conley index for discrete dynamical systems.

Received July 12, 2019
Accepted December 02, 2019

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Initially the theory of shape was introduced by Borsuk [1] as a classification of compact metric spaces which was coarser then homotopy type but which coincides with it on absolute neighborhood retracts (ANR's). His idea was to take into account the global properties of compact metric spaces and neglect the local ones. Shape can be called Čech homotopy type and its relationship to homotopy type is analogous to the relationship between Čech homology and singular homology as mentioned by Dydak – Segal in [2].

Consider the following example. Let X denote the 1-sphere S^1 and let Y denote the Warsaw circle, i.e., the graph of the function $x \to \sin\frac{1}{x}$ for $0 < x \leqslant \frac{1}{\pi}$ including the limit segment on the ordinate axis and an arc joining the endpoint of this segment to $\left(\frac{1}{\pi}, 0\right)$ which is disjoint from the graph except at its end points (see Fig. 4). Then X and Y are of different homotopy type, but will turn out to be of the same shape. These spaces fail to be of the same homotopy type because there are not enough maps (continuous functions) of X into Y due to the failure of Y to be locally connected. Since any continuous image of X must be a locally connected continuum, it must be an arc in Y and so any such map is homotopically trivial. In other words, local difficulties prevent X and Y from being of the same homotopy type even though globally they are very much alike (e.g., they both divide the plane into two components). Borsuk remedied this difficulty by introducing the notion of fundamental sequence which is more general than that of a mapping. This definition is quite appealing from the intuitive point of view, for it shows a strong resemblance with some kind of limit in the sense of homotopy (see [1]).

Let Cpt denote the category of compact metrizable spaces and continuous mappings between them. If $(P_k)_{k\in\mathbb{N}}$ is a decreasing sequence of compact spaces, i.e., $P_{k+1} \subseteq P_k$ for every $k \in \mathbb{N}$, let us call $P := \cap_{k\in\mathbb{N}} P_k$ the limit of $(P_k)_{k\in\mathbb{N}}$.

Definition 1. *Let $F\colon Cpt \to \mathcal{C}$ be a functor. We say that F is continuous if, whenever $(P_k)_{k\in\mathbb{N}}$ is a decreasing sequence of compact spaces with the property that if the inclusion $j_k\colon P_k \to P_1$ transforms into an equivalence under F, (i.e., $F(j_k)$ is an equivalence in the category $\mathcal{C}$) for every $k \in \mathbb{N}$, then the same holds for the limit inclusion, i.e., the inclusion $j\colon P \to P_1$ transforms into an equivalence under F.*

Example 1. The continuity theorem for Čech cohomology theory (see [3]) shows that the latter is a continuous functor $\check{\text{H}} : Cpt \to Ab$ with values in the category of graded abelian groups.

If we denote by $HCpt$ the category of compact spaces and homotopy classes of continuous mappings between them, and let $H\colon Cpt \to HCpt$ be the functor taking each space to itself and each continuous mapping to its homotopy class, it turns out that H is not continuous. This issue can be overcome by replacing homotopy theory by shape theory, which can be thought of as some kind of continuous (or Čech type) homotopy theory as previously mentioned.

The objects of the shape category of compact spaces $SCpt$ are the same as those in $HCpt$ and Cpt, just compact metrizable spaces. The morphisms are, however, a bit more complicated. Every continuous mapping $f\colon X \to Y$ induces a shape morphism $Sh(f)\colon X \to Y$ which depends only on the homotopy class of f, but in general there may exist shape morphisms $u\colon X \to Y$ that do not come from any continuous mapping. Thus $SCpt$ contains representatives of all the morphisms in $HCpt$ plus some extra ones which account for its flexibility when it comes to comparing its objects.

We shall say that two spaces X and Y have the same shape, and represent it by $Sh(X) = Sh(Y)$, if there exist two shape morphisms $u\colon X \to Y$ and $v\colon Y \to X$ which are inverses to each other, that is, $v \circ u = Sh(idX)$ and $u \circ v = Sh(idY)$. Both u and v are called shape equivalences.

The following theorem collects some properties of shape theory relevant to this paper.

Theorem 1. *Let $Sh\colon Cpt \to SCpt$ denote the functor which takes every compact space to itself and every continuous mapping to the shape morphism it induces.*

i) *If $u\colon X \to Y$ is a shape morphism between two spaces X and Y having the homotopy type of finite simplicial complexes (in particular, if X and Y are compact manifolds), there exists a continuous mapping $f\colon X \to Y$ such that $Sh(f) = u$.*

ii) *The shape functor Sh is continuous.*

iii) *Two spaces with the same shape have isomorphic Čech cohomology groups.*

The first assertion in the previous theorem means that homotopy and shape theory are equivalent when the underlying spaces have the homotopy type of finite simplicial complexes. Thus, shape theory, while yielding a coarser classification of spaces than homotopy theory, is still as good as it in distinguishing well-behaved spaces, but also enjoys the important continuity property.

Let us note that homotopy theory enters the scene of dynamical systems in a quite straightforward way, since flows provide a natural means of constructing homotopy equivalences (see [18] for more details). However, and in a spirit similar to that which motivated our previous discussion, we would like to be able to pass to the limit, in some sense.

Suppose that M is a positively invariant compact set. The flow induces retractions $r_t\colon M \to \Phi(M,t)$ given by $r_t(p) := \Phi(p,t)$, and homotopies $H(p,\tau) = \Phi(p,\tau), \tau \in [0,t]$ such that $id_M \simeq r_t$ for every $t \geqslant 0$. In particular, the assertion:

$$(HE)_t: \quad \text{The inclusion } j_t\colon \Phi(M,t) \to M \text{ is a homotopy equivalence}$$

is true for every $t \geqslant 0$.

Now the question is whether the same still holds on letting $t \to \infty$, that is, whether:

$$(HE)_\infty: \quad \text{The inclusion } j\colon \omega(M) = \Phi(M,+\infty) \to M \text{ is a homotopy equivalence}$$

holds true. As we mentioned previously, the homotopy functor is not continuous, but the shape functor is. So for compact metric spaces the shape category enables us to pass to the limit, and conclude that the inclusion is a shape equivalence.

The main aim of this paper is to show how the new intrinsic approach to shape, introduced by Shekutkovski in [6], can be used to study the topological structure of a global attractor compared to that of the phase space in noncompact spaces.

The main theorem of this paper holds for arbitrary metric spaces. Similar results were proved by Hastings, Bogatyi and Gutsu, Sanjurjo, Kapitanski and Rodnianski in [7, 9, 16, 17] with additional assumptions on the phase space and the flow.

A result similar to this has been established in the paper by Kapitanski and Rodnianski, but under additional assumptions of completeness and using the exterior approach to shape.

2. Intrinsic shape for paracompact spaces

Shape theory was introduced by Borsuk [1] in order to study geometric properties of compact metric spaces with not necessarily good local properties. Namely, homotopy theory turns out to be an inappropriate tool for studying spaces with local pathology which appear in the mathematical formulation of many natural phenomena, for example, solenoids, attractors etc. Hence, it is natural to look for another adequate tool for handling these problems. Shape theory takes the role in this context because it manages to smooth out local pathologies while preserving global properties. Besides, shape theory does not modify homotopy theory in the good framework.

Initially, the external approach to shape due to Borsuk requires external elements (ambient-AR, ANR-expansions) in which the original space is embedded. Later on, a question was raised regarding the intrinsic description of shape, i.e., construction without external spaces.

The first intrinsic approach to shape is given in Refs. [19] and [20]. In Ref. [15], using the notion of a proximate sequence over cofinal sequences of finite coverings, intrinsic shape category is constructed for compact metric spaces. For paracompact spaces the notion of a proximate sequence is replaced by a proximate net indexed by locally finite coverings from the set of all coverings *CovX*.

We shall follow the construction given in [6] for paracompact spaces using the notion of $\mathcal{V}$-continuity.

By a covering we understand a covering consisting of open sets and the set of all coverings is denoted by *CovX*. For technical reasons a covering containing an empty set will be considered the same with the covering without an empty set.

Let us start with some of the basic definitions:

For collections $\mathcal{U}$ and $\mathcal{V}$ of subsets of X, $\mathcal{U} \prec \mathcal{V}$ means that $\mathcal{U}$ refines $\mathcal{V}$, i.e., each $U \in \mathcal{U}$ is contained in some $V \in \mathcal{V}$.

Definition 2. *Suppose $\mathcal{V}$ is a covering of Y. A function $f\colon X \to Y$ is $\mathcal{V}$-continuous at a point $x \in X$ if there exists a neighborhood U_x of x and $V \in \mathcal{V}$, such that:*

$$f(U_x) \subseteq V.$$

A function $f\colon X \to Y$ is $\mathcal{V}$-continuous if it is $\mathcal{V}$ -continuous at every point $x \in X$. In this case, the family of all U_x forms a covering of X.

According to this, $f\colon X \to Y$ is $\mathcal{V}$-continuous if there exists a covering $\mathcal{U}$ of X such that for any $x \in X$ there exists a neighborhood U of x, and $V \in \mathcal{V}$ such that $f(U) \subseteq V$. We denote shortly: there exists $\mathcal{U}$, such that $f(\mathcal{U}) \prec \mathcal{V}$.

If $f\colon X \to Y$ is $\mathcal{V}$-continuous, then $f\colon X \to Y$ is $\mathcal{W}$-continuous for any $\mathcal{W}$ such that $\mathcal{V} \prec \mathcal{W}$.

If $\mathcal{V}$ is a covering of Y and $V \in \mathcal{V}$, the open set $\mathrm{st}(V)$ (star of V) is the union of all $W \in \mathcal{V}$ such that $W \cap V \neq \varnothing$. We form a new covering of Y, $\mathrm{st}(\mathcal{V}) = \{\mathrm{st}(V) | V \in \mathcal{V}\}$.

Definition 3. *The functions f, $g\colon X \to Y$ are $\mathcal{V}$-homotopic if there exists a function $F\colon X \times I \to Y$ such that:*

i) $F\colon X \times I \to Y$ *is* $\mathrm{st}(\mathcal{V})$*-continuous,*

ii) $F\colon X \times I \to Y$ *is $\mathcal{V}$-continuous at all points of $X \times \partial I$,*

iii) $F(x, 0) = f(x)$, $F(x, 1) = g(x)$.

The relation of $\mathcal{V}$-homotopy is denoted by $f \overset{\mathcal{V}}{\simeq} g$. This is an equivalence relation.

Definition 4. *A proximate net $(f_\mathcal{V})\colon X \to Y$ is a family $\underline{f} = (f_\mathcal{V} \mid \mathcal{V} \in CovY)$ of $\mathcal{V}$-continuous functions $f_\mathcal{V}\colon X \to Y$ such that, if $\mathcal{V} \succ \mathcal{W}$, then $f_\mathcal{V}$ and $f_\mathcal{W}$ are $\mathcal{V}$-homotopic.*

Remark 1. If $\mathcal{V}$ is a covering of Y and $M \subseteq Y$, then by $\mathcal{V} \cap M$ we denote the following covering of M:

$$\mathcal{V} \cap M = \{V \cap M | V \in \mathcal{V}\}.$$

If a proximate net $(f_{\mathcal{V}})\colon X \to Y$ satisfies $f_{\mathcal{V}}(X) \subseteq M$ and if for $\mathcal{V} \succ \mathcal{W}$, $f_{\mathcal{V}}$ and $f_{\mathcal{W}}$ are $\mathcal{V}$-homotopic in M, then we can define a proximate net $(f_{\mathcal{W}})\colon X \to M$ in the following way:

For a covering $\mathcal{W}$ of M we choose a covering $\mathcal{V}$ of Y such that $\mathcal{V} \cap M = \mathcal{W}$.

Then the function $f_{\mathcal{W}}\colon X \to M$ is defined by

$$f_{\mathcal{W}}(x) = f_{\mathcal{V}}(x), \quad \text{for } x \in X.$$

Then $(f_{\mathcal{W}})\colon X \to M$ is a proximate net.

We say that the proximate net $(f_{\mathcal{W}})\colon X \to M$ is inherited from $(f_{\mathcal{V}})\colon X \to Y$, and denote the inherited proximate net by $(f_{\mathcal{V} \cap M} | \mathcal{V} \in Cov Y)$.

Two proximate nets $(f_{\mathcal{V}})\colon X \to Y$ and $(g_{\mathcal{V}})\colon X \to Y$ are homotopic if $f_{\mathcal{V}}$ and $g_{\mathcal{V}}$ are $\mathcal{V}$-homotopic for all $\mathcal{V} \in Cov Y$, which we denote by $(f_{\mathcal{V}}) \sim (g_{\mathcal{V}})$. This is an equivalence relation.

If $(f_{\mathcal{V}})\colon X \to Y$ and $(g_{\mathcal{W}})\colon Y \to Z$ are proximate nets, then for a covering $\mathcal{W} \in Cov Z$ there exists a covering $\mathcal{V} \in Cov Y$ such that $g_{\mathcal{W}}(\mathcal{V}) \prec \mathcal{W}$. Then the composition of these two proximate nets is a proximate net $(h_{\mathcal{W}})\colon X \to Z$ defined by $(h_{\mathcal{W}}) = (g_{\mathcal{W}} f_{\mathcal{V}})\colon X \to Z$.

Paracompact spaces and homotopy classes of proximate nets $[(f_{\mathcal{V}})]$ form a category whose isomorphisms induce classifications which coincide with the standard shape classification, i.e., isomorphic spaces in this category have the same shape.

At the end of this section we give two lemmas that will be used in the sequel. The proof is given in [21] and [6].

Definition 5. *Let $\mathcal{V}$ be a covering of Y. Two functions f, $g\colon X \to Y$ are $\mathcal{V}$-near if for any $x \in X$ there exists $V \in \mathcal{V}$ such that $f(x), g(x) \in V$.*

Lemma 1. *If $\mathcal{V}$ is a covering of Y and f, $g\colon X \to Y$ are $\mathcal{V}$-near and $\mathcal{V}$-continuous, then f and g are $\mathcal{V}$-homotopic.*

Lemma 2. *Suppose $\mathcal{V}$ is a finite covering of Y, $X = X_1 \cup X_2$, X_i closed, $i = 1, 2$ and $f_i\colon X_i \to Y$, $\mathcal{V}$-continuous functions, $i = 1, 2$ such that $f_1(x) = f_2(x)$, for all $x \in X_1 \cap X_2$. We define a function by*

$$f(x) = f_i(x), \quad \textit{for } x \in X_i,\ i = 1, 2.$$

Then

1) *If $x \in Int X_1$ or $x \in Int X_2$, then $f\colon X \to Y$ is $\mathcal{V}$-continuous at x.*
2) *If $x \in \partial X_1$ or $x \in \partial X_2$, then $f\colon X \to Y$ is $\mathrm{st}(\mathcal{V})$-continuous at x.*

3. Basic notions about dynamical systems

Before proceeding further we will recall some elementary concepts from [4].

Definition 6. *Let (X, d) be a given metric space. A flow in X is a continuous map $\Phi\colon X \times \mathbb{R} \to X$ such that it satisfies the following two conditions:*

$$\Phi(x, 0) = x, \quad \Phi(\Phi(x, t), s) = \Phi(x, t + s) \quad \textit{for all } x \in X \textit{ and } t,\ s \in \mathbb{R}.$$

The triplet $(X, \mathbb{R}, \Phi)$ forms a dynamical system (flow) with phase map Φ and phase space X. If we replace the set $\mathbb{R}$ with $\mathbb{R}^+$, we get the corresponding notion of semidynamical system.

For every $t \in \mathbb{R}$ we will consider the map $\Phi_t\colon X \to X$ defined by $\Phi_t(x) = \Phi(x, t)$.

Definition 7. *We say that a given subset $M \subseteq X$ is invariant under the flow Φ if $\Phi(M,t) \subseteq M$, for all $t \in \mathbb{R}$. If we replace the set $\mathbb{R}$ with $\mathbb{R}^+$ or $\mathbb{R}^-$, we obtain the corresponding notions of positively and negatively invariant set.*

Definition 8. *A compact invariant set M is stable if every neighborhood U of M admits a positively invariant neighborhood V of M such that $V \subseteq U$.*

The trajectory of a point x is the set $\gamma(x) = \{\Phi(x,t) | t \in \mathbb{R}\}$. By replacing the set $\mathbb{R}$ with $\mathbb{R}^+ \cup \{0\}$ or $\mathbb{R}^- \cup \{0\}$ we obtain the corresponding notions of positive and negative semi-trajectory. We denote them by $\gamma^+(x)$ and $\gamma^-(x)$, respectively. We introduce the positive limit set of a given subset $M \subseteq X$ with the following:

$$\omega(M) = \{x \in X \mid \exists x_n \in M,\ t_n \to \infty,\ \Phi(x_n,t_n) \to x\}.$$

Analogously, we define the negative limit set by $\alpha(M)$. If M is an invariant set, its region of attraction $\mathcal{A}(M)$ is the set of all points $x \in X$ such that $\varnothing \neq \omega(x) \subset M$. We say that M is an attractor if $\mathcal{A}(M)$ is a neighborhood of M. Stable attractors are called asymptotically stable sets. Before introducing the concept of a global attractor, we need the following definition:

Definition 9. *A set $M \subseteq X$ attracts a set $C \subseteq X$ if for every neighborhood U of M there exists $T \in \mathbb{R}$ such that $\Phi_t(C) \subseteq U$, for every $t \geqslant T$.*

Definition 10. *Let (Φ_t) be a semidynamical system on a metric space X. A compact positively invariant set $M \subseteq X$ is said to be a global attractor if it attracts all bounded sets and is minimal with respect to this property.*

Remark 2. In the paper by Kapitanski–Rodnianski [17] there is no requirement of positive invariance. There is a result in the paper by V. Chepyzhov and M. Conti [22] that under the additional condition of completeness the attractor must be positively invariant.

The following result is known as Keesling's reformulation of Beck's theorem [24].

Theorem 2. *Let X be a metric space and let $\Phi\colon X \times \mathbb{R} \to X$ be a flow on X with S as a set of fixed points. Then for any closed set S' containing S one can construct a new flow $\Phi'\colon X \times \mathbb{R} \to X$ with S' as a set of fixed points. Moreover, for any $x \in X \backslash S'$ with trajectory $\gamma(x)$ under Φ, the trajectory of x under Φ' is just the set of points which can be joined to x by an arc in $\gamma(x) \backslash S'$.*

Remark 3. Keesling's reformulation of Beck's theorem can be seen to hold also for semidynamical systems.

4. Shape of global attractors in arbitrary metric spaces

In this paper we apply the theory of intrinsic shape to deduce a result for global attractor properties compared with the properties of the phase space in terms of shape theory.

In order to prove the main theorem, we need the following:

A covering $\mathcal{V}$ of M in X is called **regular** if it satisfies the following conditions:

1) If $V \in \mathcal{V}$, then $V \cap M \neq \varnothing$.
2) If U, $V \in \mathcal{V}$ and $U \cap V \neq \varnothing$, then $U \cap V \in \mathcal{V}$.

For a covering $\mathcal{V}$ of M we introduce the notation $|\mathcal{V}| = \bigcup_{V\in\mathcal{V}} V$.

For a finite regular covering $\mathcal{V}$ we define a function $r_\mathcal{V}\colon |\mathcal{V}| \to M$ in the following way:

For points $y \in M$ we put $r_\mathcal{V}(y) = y$

For points $y \in |\mathcal{V}|\backslash M$, by induction we can choose the smallest member $V \in \mathcal{V}$ such that $y \in V$, then choose a fixed point $y_V \in V \cap M$ and put $r_\mathcal{V}(y) = y_V$.

The function $r_\mathcal{V}$ is $\mathcal{V}$-continuous.

Lemma [21]. *If $\mathcal{V} \succ \mathcal{W}$, then $r_\mathcal{W}\colon |\mathcal{W}| \to M$ and $r_\mathcal{V} : |\mathcal{W}| \to M$ (the restriction of $r_\mathcal{V}\colon |\mathcal{V}| \to M$ to $|\mathcal{W}|$) are $\mathcal{V}$-near and so $\mathcal{V}$-homotopic by a homotopy $r_{\mathcal{V}\mathcal{W}}\colon |\mathcal{W}| \times I \to M$.*

Lemma [23]. *$i \circ r_\mathcal{V}\colon |\mathcal{V}| \to |\mathcal{V}|$ and $1_\mathcal{V}\colon |\mathcal{V}| \to |\mathcal{V}|$ are $\mathcal{V}$-homotopic by a homotopy $R_\mathcal{V}\colon |\mathcal{V}| \times I \to |\mathcal{V}|$ such that $R_\mathcal{V}(x, t) = x$, for $x \in M$.*

We proceed by showing how a semidynamical system $(\Phi_t\colon X \to X)$ in a bounded metric space X with a global attractor M induces a shape morphism $X \to M$ in a natural way.

Let us note that there exists a sequence of finite regular coverings $(\mathcal{V}_n)$ of the compact M in X which is cofinal in the set of all coverings of X.

For $\mathcal{V}_1$ we consider the set of coverings of X:

$$N_1 = \{\mathcal{V}|\mathcal{V}_1 \prec \mathcal{V}\}.$$

There exists t_1 such that $\Phi(X, (t_1, \infty)) \subseteq |\mathcal{V}_1|$. We put $t_\mathcal{V} = t_1$ for all $\mathcal{V} \in N_1$.

For $\mathcal{V}_2$ we consider the set of coverings of X:

$$N_2 = \{\mathcal{V}|\mathcal{V}_2 \prec \mathcal{V}\}\backslash N_1.$$

There exists $t_2 \geqslant t_1$ such that $\Phi(X, (t_2, \infty)) \subseteq |\mathcal{V}_2|$. We put $t_\mathcal{V} = t_2$ for all $\mathcal{V} \in N_2$.

By induction we define the set of coverings of X:

$$N_n = \{\mathcal{V}|\mathcal{V}_n \prec \mathcal{V}\}\backslash \bigcup_{i=1}^{n-1} N_i.$$

There exists $t_n \geqslant t_{n-1}$ such that $\Phi(X, (t_n, \infty)) \subseteq |\mathcal{V}_n|$. We put $t_\mathcal{V} = t_n$ for all $\mathcal{V} \in N_n$.

Hence, we have defined a net of positive real numbers $(t_\mathcal{V}|\mathcal{V} \in CovX)$ such that the following property holds:

$$t_\mathcal{V} \to \infty \quad \text{and} \quad \mathcal{V} \succ \mathcal{W} \Rightarrow t_\mathcal{W} \geqslant t_\mathcal{V}. \tag{4.1}$$

Namely, if n is minimal such that $\mathcal{V} \succ \mathcal{V}_n$ and m is minimal such that $\mathcal{W} \succ \mathcal{V}_m$, then from $\mathcal{V} \succ \mathcal{W} \succ \mathcal{V}_m$ we conclude that $m \geqslant n$. Hence, from $t_\mathcal{V} = t_n$ and $t_\mathcal{W} = t_m$ we obtain (4.1).

Construction of the proximate net

We choose an arbitrary covering $\mathcal{V}$ of X.

There exists $n \in \mathbb{N}$ such that $\mathcal{V} \succ \mathcal{V}_n$. We choose n to be minimal with respect to this property. Hence, $\mathcal{V} \in N_n$.

We mention that, since $r_{\mathcal{V}_n}, R_{\mathcal{V}_n}$ are $\mathcal{V}_n$-continuous, they will be $\mathcal{V}$-continuous.

For $x \in X$ and $t \in [t_\mathcal{V}, \infty)$ we define:

$$k_\mathcal{V}(x, t) = r_{\mathcal{V}_n}(\Phi(x, t)).$$

Note that the function is $\mathcal{V}_n$-continuous and hence $\mathcal{V}$-continuous.

Let us also mention that, using Beck's theorem 2 from the previous section for points $x \in M$, the following holds:

$$k_\mathcal{V}(x, t) = r_{\mathcal{V}_n}(\Phi(x, t)) = r_{\mathcal{V}_n}(x) = x.$$

Now we define $f_{\mathcal{V}}\colon X \to X$ by

$$f_{\mathcal{V}}(x) = r_{\mathcal{V}_n}(\Phi(x, t_{\mathcal{V}})) = r_{\mathcal{V}_n}(\Phi(x, t_n)).$$

The composition is well defined (see Fig. 1). Since $\mathcal{V} \succ \mathcal{V}_n$, the function $f_{\mathcal{V}}$ is $\mathcal{V}$-continuous.

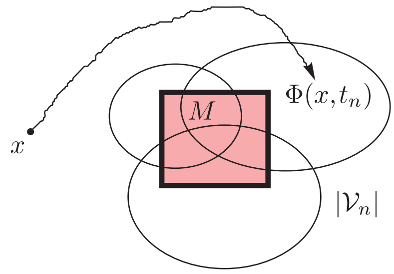

Fig. 1. The composition of $f_{\mathcal{V}}$.

In this way the required proximate net $(f_{\mathcal{V}})$ is defined such that $f_{\mathcal{V}}(X) \subseteq M$.

We are ready to state our main theorem:

Theorem 3. *Let X be an arbitrary metric space and let $(\Phi_t\colon X \to X \mid t \in \mathbb{R}^+)$ be a semi-dynamical system with a global attractor M. Then the inclusion $i\colon M \to X$ induces a shape equivalence.*

Proof. We will prove the theorem in two stages.

(i) First we will assume that the metric space X is bounded. Then we can use the previous construction of a family of functions $f_{\mathcal{V}}\colon X \to X$ with $f_{\mathcal{V}}(X) \subseteq M$ to formulate the following:

Lemma 3. *The net of functions $f_{\mathcal{V}}\colon X \to X$ is a proximate net.*

Proof. We need to prove that, if $\mathcal{V} \succ \mathcal{W}$, then $f_{\mathcal{V}}$ and $f_{\mathcal{W}}$ are $\mathcal{V}$-homotopic. We will define a homotopy $r_{\mathcal{V}\mathcal{W}}\colon X \times I \to X$ in the following manner:

We define $R_{\mathcal{V}\mathcal{W}}\colon X \times I \to X$ by $R_{\mathcal{V}\mathcal{W}}(x,t) = k_{\mathcal{V}}(x, (1-t)t_{\mathcal{V}} + tt_{\mathcal{W}})$. Then $R_{\mathcal{V}\mathcal{W}}(x,0) = f_{\mathcal{V}}(x)$ and $R_{\mathcal{V}\mathcal{W}}(x,1) = k_{\mathcal{V}}(x, t_{\mathcal{W}})$. From $k_{\mathcal{V}}(x,t_{\mathcal{W}}) = r_{\mathcal{V}}\Phi(x,t_{\mathcal{W}})$ and $k_{\mathcal{W}}(x,t_{\mathcal{W}}) = r_{\mathcal{W}}\Phi(x,t_{\mathcal{W}})$ it follows that the two functions $f_{\mathcal{W}}, l\colon X \to X$ defined by

$$l(x) = k_{\mathcal{V}}(x, t_{\mathcal{W}}) \quad \text{and} \quad f_{\mathcal{W}}(x) = k_{\mathcal{W}}(x, t_{\mathcal{W}})$$

are $\mathcal{V} \cap M$-near and by Lemma 1 are $\mathcal{V} \cap M$-homotopic, say by a homotopy $h_{\mathcal{V}}$. By $\mathcal{V} \cap M$ we denote shortly the covering of M defined by $\mathcal{V} \cap M = \{V \cap M \mid V \in \mathcal{V}\}$ (see Remark 1). Then the concatenation of the homotopies $r_{\mathcal{V}\mathcal{W}} = R_{\mathcal{V}\mathcal{W}} * h_{\mathcal{V}}$ is the required $\mathcal{V}$-homotopy satisfying $r_{\mathcal{V}\mathcal{W}}(x,0) = f_{\mathcal{V}}(x)$ and $r_{\mathcal{V}\mathcal{W}}(x,1) = f_{\mathcal{W}}(x)$. □

Lemma 4. *Suppose X is a metric space and M a compact subset of X. If there is a proximate net $(f_{\mathcal{V}})\colon X \to X$ such that $f_{\mathcal{V}}(X) \subseteq M$, for all coverings $\mathcal{V}$, and for $\mathcal{V} \succ \mathcal{W}$, the functions $f_{\mathcal{V}}$ and $f_{\mathcal{W}}$ are $\mathcal{V}$-homotopic in M and there is a homotopy $H_{\mathcal{V}}\colon X \times I \to X$ such that:*

1) $H_{\mathcal{V}}(x,0) = x, H_{\mathcal{V}}(x,1) = f_{\mathcal{V}}(x)$.
2) $H_{\mathcal{V}}(M \times I) \subseteq M$.

Then the inherited proximate net $(f_{\mathcal{V}\cap M})\colon X \to M$ induces a shape equivalence with the inclusion $i\colon M \to X$ as a shape inverse.

Proof. We consider the proximate net $(i_{\mathcal{V}})\colon M \to X$ induced by the inclusion $i\colon M \to X$. First, $i_{\mathcal{V}} \circ f_{\mathcal{V}\cap M}(x) = f_{\mathcal{V}}(x)$ and by 1) , $H_{\mathcal{V}}$ connects $i_{\mathcal{V}} \circ f_{\mathcal{V}\cap M}$ and the identity map 1_X. On the other hand, $f_{\mathcal{V}\cap M} \circ i_{\mathcal{V}}(x) = f_{\mathcal{V}\cap M}(x)$ and by 2), $H_{\mathcal{V}}|_M$ connects $f_{\mathcal{V}\cap M} \circ i_{\mathcal{V}}$ and the identity map 1_M. □

Proposition 1. *The shape morphism* $[(f_{\mathcal{V}\cap M})]\colon X \to M$ *is a shape equivalence. Consequently,* $Sh(M) = Sh(X)$.

Proof. We choose an arbitrary covering $\mathcal{V}$ of X. As mentioned above, during the definition of the proximate net $(f_{\mathcal{V}})$ there exists a sequence of regular and finite coverings $(\mathcal{V}_n)$ of M in X which is cofinal in the set of all coverings of X. Hence, there exists $n \in \mathbb{N}$ such that $\mathcal{V} \succ \mathcal{V}_n$. We choose the natural number n to be minimal with respect to this property. For simplicity we put $\mathcal{V}_n = \mathcal{W}$.

We mention that, since $r_{\mathcal{W}}, R_{\mathcal{W}}$ are $\mathcal{W}$-continuous, they will be $\mathcal{V}$-continuous as well. We will define a homotopy $H_{\mathcal{V}}\colon X \times I \to X$ as a concatenation of three homotopies. The first is a continuous map $F\colon X \times I \to X$, defined by

$$F(x, s) = \Phi(x, st_{\mathcal{V}}).$$

This map satisfies

$$F(x, 0) = x, \quad F(x, 1) = \Phi(x, t_{\mathcal{V}}).$$

The third is a function $G\colon X \times I \to X$ defined by

$$G(x, s) = r_{\mathcal{W}}\Phi(x, t_{\mathcal{V}}).$$

The composition $r_{\mathcal{W}}\Phi$ is $\mathcal{V}$-continuous and this map satisfies

$$G(x, 0) = r_{\mathcal{W}}\Phi(x, t_{\mathcal{V}}), \quad G(x, 1) = r_{\mathcal{W}}\Phi(x, t_{\mathcal{V}}) = f_{\mathcal{V}}(x).$$

The middle homotopy $Q\colon X \times I \to X$ is defined by

$$Q(x, s) = R_{\mathcal{W}}(\Phi(x, t_{\mathcal{V}}), s),$$

where $R_{\mathcal{W}}$ is the homotopy from Lemma [23]. This homotopy satisfies

$$Q(x, 0) = F(x, 1) = \Phi(x, t_{\mathcal{V}}), \quad Q(x, 1) = G(x, 0) = r_{\mathcal{W}}\Phi(x, t_{\mathcal{V}})$$

since

$$F(x, 1) = Q(x, 0) \quad \text{and} \quad Q(x, 1) = G(x, 0).$$

We can define the concatenation of the three defined homotopies and finally define the required homotopy $H_{\mathcal{V}}\colon X \times I \to X$ by $H_{\mathcal{V}} = G * Q * F$ and

$$H_{\mathcal{V}}(x, 0) = x, \quad H_{\mathcal{V}}(x, 1) = f_{\mathcal{V}}(x).$$

Since $F(x, s) = x$ for $x \in M$ and the same holds for Q and G, we deduce that the same holds for the homotopy $H_{\mathcal{V}}$, i.e., $H_{\mathcal{V}}(x, s) = x$, for any $x \in M$. □

(ii) Now let us discuss the second case. We assume that the metric space X is unbounded. Using the standard technique of primitive Lyapunov functions, we get a deformation retract from X to a bounded and positively invariant neighborhood K of M.

Namely, let $\tau(x) = \sup_{t\geqslant 0} dist(\Phi(x,t), M)$. We define the function

$$L(x) = \int_0^\infty \exp^{-t} \tau(\Phi(x,t))\, dt$$

which is a primitive Lyapunov function for the semidynamical system $(X, \mathbb{R}^+, \Phi)$. We choose $0 < \epsilon < \sup_{x\in X} L(x)$ and define the set

$$K = \{x \in X |\, L(x) \leqslant \epsilon\}$$

which is positively invariant and bounded. Moreover, the set M is a global attractor for the semidynamical system $(K, \mathbb{R}^+, \Phi)$. Hence, the inclusion $i\colon M \to K$ is a shape equivalence. For arbitrary $x \in X \setminus K$ there is a unique $t = t_x$ such that $L(\Phi(x, t_x)) = \epsilon$.

We define the following map:

$$m(x) = \begin{cases} t_x, & \text{if } x \in X \setminus K, \\ 0, & \text{otherwise.} \end{cases}$$

(The map is continuous). Finally, we define a map $r\colon X \to K$ by

$$r(x) = \Phi(x, m(x)).$$

Hence, K is a deformation retract of X. □

Example 2. The Warsaw circle, i.e., the graph of the function $x \to \sin\frac{1}{x}$ for $0 < x \leqslant \frac{1}{\pi}$ including the limit segment on the ordinate axis and an arc joining the endpoint of this segment to $\left(\frac{1}{\pi}, 0\right)$ is an attractor of a flow on $\mathbb{R}^2$, as mentioned by Hastings [7, Example 3.3]. But according to our theorem, the Warsaw circle cannot be a global attractor for any flow on the plane, having nontrivial shape.

Example 3. Consider a three-dimensional solid torus $P \subseteq \mathbb{R}^3$ and a necklace $M \subseteq int\, P$ consisting of infinitely many solid balls, tangent to each other and centered at the core circumference of P, which we shall denote by L. It is not difficult to construct a differentiable flow Φ which enters P transversally through its boundary ∂P and has M as a stable attractor. If we define a semiflow $\{\Phi_t |\, t \in \mathbb{R}^+\}$ on P using the flow Φ, it is easily seen that M will be a global attractor for the semidynamical system (P, Φ_t). Hence, according to our theorem, $Sh(M) = Sh(P)$.

5. Applications of intrinsic shape in dynamical systems

The first papers concerning intrinsic shape were focused on proving the equivalence with the original Borsuk approach.

The papers [8, 21], and [23] contain explicit applications of intrinsic shape from [15] to dynamical systems and show the advantage of the intrinsic approach to shape when dealing with dynamical systems.

The papers contain new theoretical approaches and new methodologies based on the use of intrinsic shape to dynamical systems.

In [21], for flows defined on a compact manifolds with or without boundary, it is shown that the connectivity components of a chain recurrent set possess a stronger connectivity known as joinability (or pointed 1-movability in the sense of Borsuk). As a consequence, the Van Dantzig – Vietoris solenoid cannot be a component of the chain recurrent set, although the solenoid appears as a minimal set of a flow [5].

Theorem 4. *Suppose φ is a flow on a compact manifold (with or without boundary). Then the components of the chain recurrent set are joinable.*

Example 4. By the last theorem, a Hawaiian earring (Fig. 2) can be a component of the chain recurrent set (Example 6). This is not the case with the solenoid (Fig. 3) since it is not joinable.

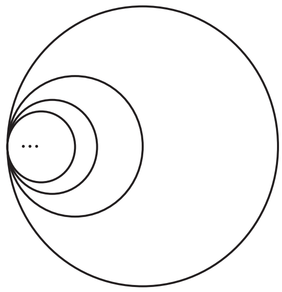

Fig. 2. Hawaiian earring.

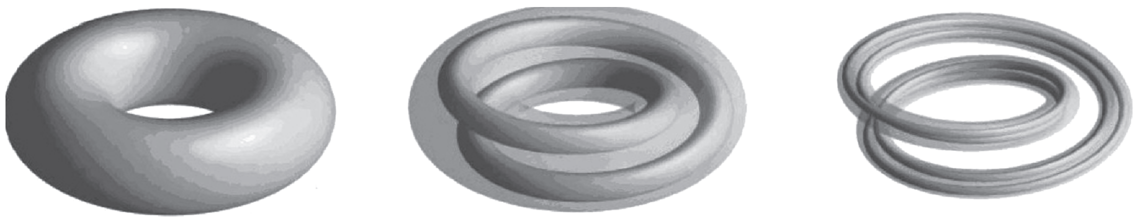

Fig. 3. T_2 twice twisted and embedded in T_1.

Example 5. The Van Dantzig – Vietoris solenoid is the intersection of members of the sequence $\mathbb{T}_1 \supseteq \mathbb{T}_2 \supseteq \ldots \supseteq \mathbb{T}_n \ldots$. The first member of the sequence $\mathbb{T}_1$ is a solid torus. Each next member of this sequence, $\mathbb{T}_i$, is a solid torus twice twisted and embedded in the previous $\mathbb{T}_{i-1}$.

It is obvious by the Lebesgue lemma that for any sequence of finite coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \ldots$, between two points of the solenoid there is a sequence of $\mathcal{V}_n$-continuous paths (k_n), $k_n\colon [0,1] \to Y$. However, there is no proximate path between two points from different arc components (see [21] and [6] for more details).

In [10], using the concept of topological transitivity, Günther and Segal raised problems involving the characterization of topologically transitive attractors on arbitrary manifolds for continuous and discrete dynamical systems.

In [25] V. Jiménez López and J. Libre gave a nice topological characterization of the ω-limit set for analytic plane flows. Using this result, a partial answer to the problems involving the characterization of topologically transitive attractors for plane flows is given in [8]. Also, a new proof of the well-known theorem [10] is presented which provides a very elegant characterization of attractors on a topological manifold:

Theorem 5. *Every attractor of a flow on a manifold has the shape of a finite polyhedron.*

The proof is based on the Conley index theory and the new intrinsic shape methods from [15].

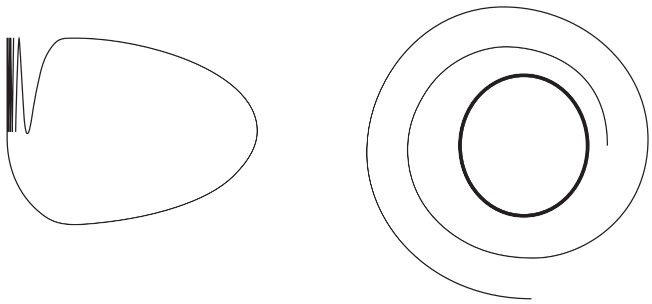

Fig. 4. Warsaw circle and circle with an approaching orbit.

By shape theory, both the Warsaw circle (Fig. 4) and a circle with an approaching orbit (Fig. 4) have the shape of a circle — a finite topological polyhedron (see Corollary 1 of our theorem). So, according to the characterization from [10], the Warsaw circle can be realized as an attractor for a suitably defined plane flow. A nice example of this fact is given by Hastings in [7].

On the other hand, the Hawaiian earring (Fig. 2), being a union of infinite many circles, is not a finite polyhedron and could not be an attractor. The solenoid (Fig. 3) is not a finite polyhedron either.

Example 6. Consider a dynamical system defined in the cylinder $D \times I$, where D stands for the unit disk. The points in the Hawaiian earring $H = \bigcup_{n=1}^{\infty} S((1/2n, 0, 1/2), 1/2n)$ are stationary points. All the points in $D \times \{0, 1\}$ are also stationary. The trajectories of the rest of the points are vertical straight lines joining two stationary points. The Hawaiian earring is a Morse set (hence a component of the chain recurrent set) which is not a shape retract of any of its neighborhoods. This shows that Theorem 3 does not hold in general for Morse sets.

Corollary 1. *Let M be a compact minimal set of the flow $\Phi\colon X \times \mathbb{R} \to X$. Then for every $x \in \mathcal{A}(M)$, $Sh(\overline{\gamma^+(x)}) = Sh(M)$.*

Proof. If $x \in M$, then $\overline{\gamma^+(x)} = M$ and the claim follows. If $x \notin M$, then since $x \in \mathcal{A}(M)$, we have that $\omega(x) \subseteq M$ and by the minimality of M, $\omega(x) = M$. Moreover, x is not periodic since $x \notin \omega(x)$. Hence, $\overline{\gamma^+(x)} = \gamma^+(x) \cup M$ with $\gamma^+(x) \cap M = \varnothing$. Now we can define a semidynamical system φ on $\gamma^+(x) \cup M$ induced by Φ. Let us note that M is a global attractor for φ. Hence, the claim follows from Theorem 3. □

Remark 4. Let us note that in general M and $\overline{\gamma^+(x)}$ are not of the same homotopy type. For example, let us consider the following system of differential equations in polar coordinates:

$$\dot{r} = r(1 - r),$$
$$\dot{\theta} = 1.$$

The unit circle M is a minimal compact set. If x is a point in $\mathcal{A}(M)$ with $x \notin M$, then $\overline{\gamma^+(x)}$ is the union of M and a ray spiralling to M (Figure 4). Obviously, $\overline{\gamma^+(x)}$ and M are not homotopically equivalent.

Corollary 2. *Let $\Phi\colon X \times \mathbb{R} \to X$ be a flow on a metric space X and let M be an asymptotically stable attractor of Φ. Then the inclusion $i\colon M \to \mathcal{A}(M)$ is a shape equivalence.*

Proof. Let us note that the region or basin of attraction $\mathcal{A}(M)$ is a positively invariant set. Hence, we can define a semiflow φ on $\mathcal{A}(M)$ induced by Φ. The set M is a global attractor for φ. The claim follows from Theorem 3. □

Another interesting corollary of Theorem 3 is the following claim.

Corollary 3. *Let $\Phi_\lambda\colon X \times \mathbb{R} \to X$, $\lambda \in I$ be a parametrized family of flows defined on a locally compact ANR, X. If M is an attractor of Φ_0, then for every neighborhood V of M contained in the basin of attraction of M there exists a λ_0, with $0 < \lambda_0 \leqslant 1$, such that for every $\lambda \leqslant \lambda_0$ there exists an attractor $M_\lambda \subset V$ of the flow Φ_λ with $Sh(M_\lambda) = Sh(M)$. Moreover, V is contained in the basin of attraction of M_λ.*

Example 7. Consider the family of ordinary differential equations defined on the plane

$$\dot{r} = -r(r^2 - \lambda),$$
$$\dot{\theta} = 1.$$

The picture on the left in Fig. 2 describes the phase portrait of the above equations when the parameter $\lambda = 0$. We see that in this case the origin is a globally attracting fixed point and the orbit of any other point spirals towards it.

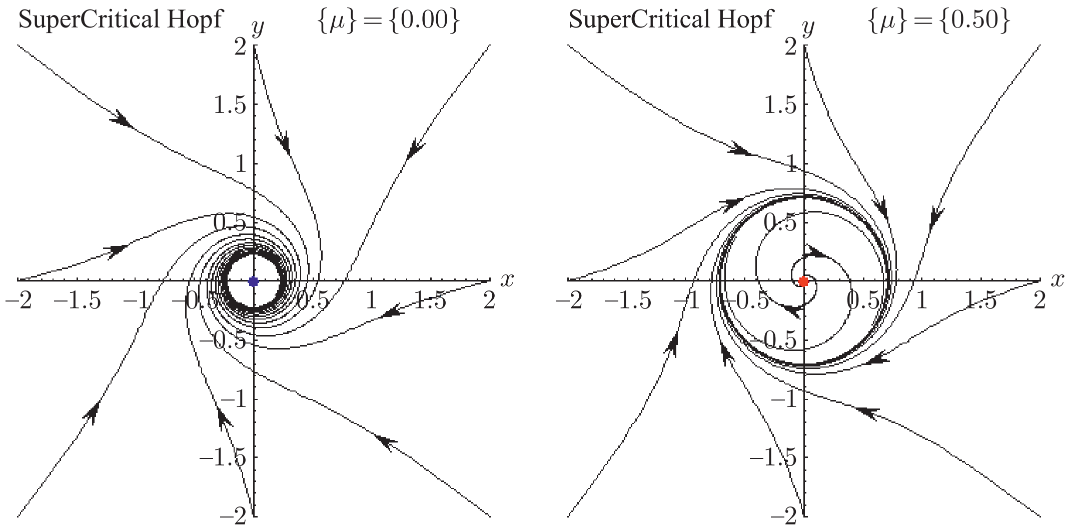

Fig. 5. Phase portrait of the family of equations.

The picture on the right describes the phase portrait of the above equations when $\lambda > 0$. In this case we see that the origin is not a global attractor anymore since, for each $\lambda > 0$, the circle centered at the origin and radius $\sqrt{\lambda}$ is a periodic trajectory which attracts uniformly all the points of the unbounded component of its complement and the origin repels all the points of the bounded one (unstable spiral and a stable limit cycle are born, hence we have a supercritical Hopf bifurcation). Notice that nevertheless for every neighborhood V of $M = (0, 0)$ there exists λ_0 with $0 < \lambda_0 \leqslant 1$, such that for every $0 < \lambda \leqslant \lambda_0$ the disk M_λ centered at the origin and radius $\sqrt{\lambda}$ is a global attractor contained in V with the shape of a point. Hence, $Sh(M_\lambda) = Sh(M)$.

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