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A dynamical approach to shape

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Abstract. In this paper we will discuss a dynamical approach to an open problem from shape theory. We will address the problem in compact metric spaces using the notion of Lebesgue number for a covering and the intrinsic approach to strong shape.

Анотація. В роботі обговорюється підхід з точки зору динамічних систем до однієї відкритої проблеми теорії шейпів. Ця проблема вивчається для компактних метричних просторів з використанням поняття числа Лебега покриття та внутрішнього підходу до сильних шейпів.

1. Introduction

The purpose of shape theory is the same as that of homotopy theory. However, for investigation of spaces that are not locally nice-like many objects that appear in dynamical systems, shape theory is a more convenient tool. The ability to smooth out local pathology makes it a good instrument for the study of the global properties of spaces which have very complicated local topological behavior. Besides, shape theory does not modify homotopy theory in the good framework. The use of shape theory in the study of dynamical systems was initiated by Hastings in [12]. He proved a kind of generalized Poincare-Bendixson theorem in higher dimensions by replacing a geometric description of an invariant set M by a description of its shape. The result of Hastings was the first one of a series of papers by different authors (for example [11,19]) who analyzed similar situations. In these papers

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useful connection has been established between shape theory and topological dynamics based on the approach to shape developed by K. Borsuk [3]. Ideas from shape theory were also important in the investigations in finite-dimensional dynamics on global attractors in [18]. Using shape theoretical methods and the Conley index an important results about robustness of global attractors were established in [17]. In the paper [8] the authors established a new connection between shape and dynamics by adopting a different point of view. The theory of dynamical systems is used here to give a new interpretation of shape.

The aim of this paper is to show how the theory of dynamical systems can be used as a tool for shape theoretical problems. The main motivation comes from an open problem in topology ([23, Problem 741]) which plunges into the question: Is every shape equivalence a strong shape equivalence? We will give a dynamical perspective of this problem for compact metric spaces using the intrinsic approach to strong shape from [21] which combines continuity up to a covering and the corresponding homotopies of second order. We will only use the notion of a Lebesgue number for a covering.

2. Attractors in metric spaces

The main reference for the elementary concepts of dynamical systems will be [1] but we also recommend [14–16]. Let us recall some of the basic notions from this theory. Let X be a metric space. A *flow* in X is a continuous map $\Phi : X \times \mathbb{R} \to X$ such that satisfies the following two conditions $\Phi(x, 0) = x$, $\Phi(\Phi(x, t), s) = \Phi(x, t + s)$ for all $x \in X$ and $t, s \in \mathbb{R}$.

The triplet $(X, \mathbb{R}, \Phi)$ forms a dynamical system (flow) with phase map Φ and phase space X. If we replace the set $\mathbb{R}$ with $\mathbb{R}^+$ we get the corresponding notion of a semi-dynamical system. The *trajectory* of a point x is the set $\gamma(x) = \{\Phi(x, t) \mid t \in \mathbb{R}\}$. By replacing the set $\mathbb{R}$ with $\mathbb{R}^+ \cup \{0\}$ or $\mathbb{R}^- \cup \{0\}$ we obtain the corresponding notions of positive and negative semi trajectory. We denote by $\gamma^+(x)$ and $\gamma^-(x)$ correspondingly. For every $t \in \mathbb{R}$ we will consider the map $\Phi_t : X \to X$ defined by $\Phi_t(x) = \Phi(x, t)$.

Definition 2.1. *We say that a given subset $M \subseteq X$ is* invariant *under the flow Φ whenever $\Phi(M, t) \subseteq M$, for all $t \in \mathbb{R}$. If we replace the set $\mathbb{R}$ with $\mathbb{R}^+$ or $\mathbb{R}^-$, we obtain the corresponding notions of* positively *and* negatively *invariant set.*

We introduce *positive limit set* of a given subset $M \subseteq X$ with the following:

$$\omega(M) = \{x \in X \mid \exists x_n \in M,\ t_n \to \infty,\ \Phi(x_n, t_n) \to x\}.$$

Similarly we define *negative limit set* $\alpha(M)$. If M is an invariant set, its *region of attraction* $\mathcal{A}(M)$ is the set of all points $x \in X$ such that $\varnothing \neq \omega(x) \subset M$. We say that M is an *attractor* if $\mathcal{A}(M)$ is a neighborhood of M.

Before introducing the concept of a global attractor we need the following definition:

Definition 2.2. *A set $M \subseteq X$ attracts a set $C \subseteq X$ if for every neighborhood U of M there exists $T \in \mathbb{R}$ such that $\Phi_t(C) \subseteq U$, for every $t \geqslant T$.*

Definition 2.3. *Let (Φ_t) be a semi-dynamical system on a metric space X. A compact positively invariant set $M \subseteq X$ is said to be a global attractor if it attracts all compact sets.*

3. Intrinsic shape and strong shape

Shape theory was introduced by Borsuk, [3, 5], in order to study geometric properties of compact metric spaces with not necessarily good local properties. Namely, homotopy theory turns out to be inappropriate tool for studying spaces with local pathology which appear in the mathematical formulation of many natural phenomena, for example solenoids, attractors etc. Hence, it is natural to look for another adequate tool for handling these problems. Shape theory takes the role in this context because manages to smooth out local pathologies while preserving global properties. Also, shape theory does not modify homotopy theory in the good framework. Initially, the external approach to shape from Borsuk requires external elements (ambient-AR, ANR-expansions) in which the original space is embedded. Later on, a question was raised regarding the intrinsic description of shape, i.e., construction without external spaces.

The first intrinsic approach to shape is given in the papers [6] and [20]. In the paper [21] using the notion of a proximate sequence over cofinal sequences of finite coverings intrinsic strong shape category is constructed for compact metric spaces.

We shall follow the construction given in [21] for compact metric spaces using the notion of $\mathcal{V}$-continuity. By a covering we understand a covering consisting of open sets. Let's start with some of the basic definitions:

For collections $\mathcal{U}$ and $\mathcal{V}$ of subsets of X, $\mathcal{U} < \mathcal{V}$ means that $\mathcal{U}$ refines $\mathcal{V}$, i.e., each $U \in \mathcal{U}$ is contained in some $V \in \mathcal{V}$.

Definition 3.1. *Suppose $\mathcal{V}$ is a covering of Y. A function $f : X \to Y$ is $\mathcal{V}$-continuous at a point $x \in X$, if there exists a neighborhood U_x of x and $V \in \mathcal{V}$ such that: $f(U_x) \subseteq V$.*

A function $f : X \to Y$ is $\mathcal{V}$-continuous if it is $\mathcal{V}$-continuous at every point $x \in X$. In this case, the family of all U_x form a covering of X.

According to this $f : X \to Y$ is $\mathcal{V}$-continuous if there exists a covering $\mathcal{U}$ of X such that for any $x \in X$, there exists a neighborhood U of x and $V \in \mathcal{V}$ such that $f(U) \subseteq V$. We denote shortly: there exists $\mathcal{U}$ such that $f(\mathcal{U}) \prec \mathcal{V}$.

If $f : X \to Y$ is $\mathcal{V}$-continuous, then $f : X \to Y$ is $\mathcal{W}$-continuous for any $\mathcal{W}$ such that $\mathcal{V} \prec \mathcal{W}$.

If $\mathcal{V}$ is a covering of Y and $V \in \mathcal{V}$ the open set $\mathrm{st}(V)$ (star of V) is the union of all $W \in \mathcal{V}$ such that $W \cap V \neq \emptyset$. We form a new covering of Y, $\mathrm{st}(\mathcal{V}) = \{\mathrm{st}(V) \mid V \in \mathcal{V}\}$.

Definition 3.2. *Two maps $f, g : X \to Y$ are $\mathcal{V}$-homotopic, if there exists a map $F : X \times I \to Y$ such that:*

(i) *$F : X \times I \to Y$ is $\mathrm{st}(\mathcal{V})$-continuous;*
(ii) *$F : X \times I \to Y$ is $\mathcal{V}$-continuous at all points of $X \times \partial I$;*
(iii) *$F(x, 0) = f(x)$, $F(x, 1) = g(x)$.*

The relation of $\mathcal{V}$-homotopy is denoted by $f \overset{\mathcal{V}}{\simeq} g$. This is an equivalence relation.

3.3. **Intrinsic shape.** We will give a brief construction of the shape category using the intrinsic approach from [21]. The approach follows the main steps of construction of homotopy theory.

Instead of taking continuous functions $f, g : X \to Y$ we take proximate sequences of functions $(f_n), (g_n) : X \to Y$ and define a relation of homotopy $(f_n) \simeq (g_n)$.

Like in homotopy theory where X and Y have the same homotopy type if there exist a continuous function $f : X \to Y$ with a homotopy inverse, X and Y have the same (intrinsic) shape if there exist a proximate sequence $(f_n) : X \to Y$ with a homotopy inverse.

Definition 3.4. *A sequence (f_n) of functions $f_n : X \to Y$ is a proximate sequence from X to Y, if for some sequence $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \ldots$ cofinal in the set of finite coverings, for all indices $m \geqslant n$, f_n and f_m are homotopic as $\mathcal{V}_n$-continuous functions, ("cofinal" means that for any finite covering $\mathcal{V}$ there exists $\mathcal{V}_n$ such that $\mathcal{V}_n \prec \mathcal{V}$). In this case we say that (f_n) is a proximate sequence over $(\mathcal{V}_n)$.*

We mention that if (f_n) and (f'_n) are proximate sequences from X to Y, then there exists a sequence $(\mathcal{V}_n)$ of finite coverings such that (f_n) and (f'_n) are proximate sequences over $(\mathcal{V}_n)$.

Two proximate sequences (f_n), (f'_n) are *homotopic* if for some sequence $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \dots$ cofinal in the set of finite coverings, (f_n) and (f'_n) are proximate sequences over $(\mathcal{V}_n)$, and for all integers n, f_n and f'_n are homotopic as $\mathcal{V}_n$-continuous functions.

This is an equivalence relation and we denote it by $(f_n) \simeq (f'_n)$. The homotopy class is denoted by $[(f_n)]$.

Let $(f_k) : X \to Y$ be a proximate sequence over $(\mathcal{V}_k)$ and let $(g_n) : Y \to Z$ be a proximate sequence over $(\mathcal{W}_n)$. For a covering $\mathcal{W}_n$ of Z, there exist a covering $\mathcal{V}_{k_n}$ of Y such that $g_n(\mathcal{V}_{k_n}) \prec \mathcal{W}_n$. Then the composition of these two proximate sequences is the proximate sequence

$$(h_n) = (g_n f_{k_n}) : X \to Z.$$

This proximate sequence is unique up to homotopy.

Compact metric spaces and homotopy classes of proximate sequences form the category whose isomorphisms induce classification which coincide with the standard shape classification, i.e., isomorphic spaces in this category have the same shape.

3.5. **Intrinsic strong shape.** The main notion for the intrinsic definition of strong shape for compact metric spaces is the notion of a strong proximate sequence.

Definition 3.6. *The sequence of pairs $(f_n, f_{n,n+1})$ of functions $f_n : X \to Y$ and $f_{n,n+1} : X \times I \to Y$ is a* strong proximate sequence from X to Y*, if there exists a cofinal sequence of finite coverings, $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \dots$ of Y such that for each natural number n, $f_n : X \to Y$ is a $\mathcal{V}_n$-continuous function and $f_{n,n+1} : X \times I \to Y$ is a homotopy connecting $\mathcal{V}_n$-continuous functions $f_n : X \to Y$ and $f_{n+1} : X \to Y$. We say that $(f_n, f_{n,n+1})$ is a* strong proximate sequence over *$(\mathcal{V}_n)$.*

If $(f_n, f_{n,n+1})$ and $(f'_n, f'_{n,n+1})$ are strong proximate sequences from X to Y, then there exists a cofinal sequence of finite coverings $(\mathcal{V}_n)$ such that $(f_n, f_{n,n+1})$ and $(f'_n, f'_{n,n+1})$ are strong proximate sequences over $(\mathcal{V}_n)$.

Two strong proximate sequences

$$(f_n, f_{n,n+1}) : X \to Y, \qquad (f'_n, f'_{n,n+1}) : X \to Y$$

are *homotopic* if there exists a strong proximate sequence

$$(F_n, F_{n,n+1}) : X \times I \to Y$$

over $\mathrm{st}(\mathcal{V}_n)$ such that:

(i) $F_n : X \times I \to Y$ is a $\mathcal{V}_n$-homotopy between $\mathcal{V}_n$-continuous maps f_n and f'_n,

(ii) $F_{n,n+1} : X \times I \times I \to Y$ is a $\mathrm{st}^2(\mathcal{V}_n)$-continuous function, at all points from $X \times \partial I^2$ is $\mathrm{st}(\mathcal{V}_n)$-continuous, $F_{n,n+1}(x,t,0) = f_{n,n+1}(x,t)$, and $F_{n,n+1}(x,t,1) = f'_{n,n+1}(x,t)$.

The composition of strong proximate sequences is defined in the following way.

Let $(f_n, f_{n,n+1}) : X \to Y$ and $(g_k, g_{k,k+1}) : Y \to Z$ be strong proximate sequences. There exists a subsequence n_k such that the sequence of pairs of functions $(h_k, h_{k,k+1})$ defined as follows is a strong proximate sequence (see [21]).

We put $h_k = g_k f_{n_k}$ and

$$h_{k,k+1}(x,t) = \begin{cases} g_k f_{n_k,n_{k+1}}(x,2t), & t \in [0;\frac{1}{2}], \\ g_{k,k+1}(f_{n_{k+1}}(x), 2t-1), & t \in [\frac{1}{2};1]. \end{cases}$$

where $f_{n_k,n_{k+1}} = f_{n_k,n_k+1} * \cdots * f_{n_{k+1}-1,n_{k+1}}$ is the concatenation of homotopies. The composition of strong proximate sequence is unique up to homotopy (see [21]).

Homotopy classes of strong proximate sequences are morphisms of strong shape category. Two spaces X and Y have the *same strong shape*, if they are isomorphic in this category and we denote it by $SSh(X) = SSh(Y)$.

4. Intrinsic strong shape of attractors in compact metric spaces

Invariant sets are of crucial importance in the theory of dynamical systems. This is because that for a given dynamical system, they are the carriers of much information on the longtime behavior of the system. Of special interest are attractors. An attractor, if exists, is the depository of "all" the dynamics of a system near the attractor. The attractor theories in metric spaces were fully developed in the past decades for both autonomous and non-autonomous systems. Here we are interested in the shape theoretical insight of these rather intriguing objects.

Several authors have established interesting results, [2,7,9,11,13], where properties of attractors are compared with properties of the phase space in terms of shape theory. It is a well known result that the inclusion $i : M \to X$ from a global attractor to its phase space is a shape equivalence. In order to build our dynamical perspective to the topological problem from [23] first we shall prove that the inclusion $i : M \to X$ induces a strong shape equivalence from a global attractor to its phase space. Also avoiding embedding in suitable AR-spaces (by use of Kuratowski-Wojdyslawski theorem) using the intrinsic approach to strong shape we shall simplify our discussion only in terms of non-continuous topology.

We start by showing how a semi-dynamical system in a compact metric space with a global attractor M induces a strong proximate sequence and hence a strong shape morphism in a natural way.

4.1. **The strong shape morphism induced by a semi-dynamical system.** Let X be a compact metric space and $\{\Phi_t : X \to X \mid t \in \mathbb{R}^+\}$ a semi-flow with a global attractor M. We will induce a strong proximate sequence $(g_n, g_{n,n+1}) : X \to M$ in a natural way using the semi-flow Φ defined in X. Namely, first we will define a map $a : X \times \mathbb{R}^+ \to M$ in the following way:

For arbitrary $x \in X$ and $t \in \mathbb{R}^+$ we go with the semi-flow until the point $\Phi(x,t)$. Then we measure the distance $d(\Phi(x,t), M)$ which by compactness of the set M is achieved in some point $m_x^t \in M$, that is

$$d(\Phi(x,t), M) := d(\Phi(x,t), m_x^t).$$

Of course this point may not be unique but nevertheless we can pick any such point. So we define $a(x,t) = m_x^t$, for $(x,t) \in X \times \mathbb{R}^+$. Note that for $x \in M$ and $t \in \mathbb{R}^+$ the following holds $a(x,t) = \Phi(x,t)$.

Now let us choose an arbitrary cofinal sequence of finite coverings $(\mathcal{V}_n)$ for the compact set M and adjoin the corresponding sequence of Lebesgue numbers (λ_n) for the coverings $(\mathcal{V}_n)$. We will define a local base for M in the form

$$\left\{\mathcal{W}_n = T\left(M, \tfrac{\lambda_n}{8}\right) \mid n \in \mathbb{N}\right\},$$

where $T\left(M, \frac{\lambda_n}{8}\right)$ stands for an open ball centered at M and radius $r = \frac{\lambda_n}{8}$. Using the fact that M is a global attractor there exists a monotonically increasing sequence of real numbers $t_n \to +\infty$ such that $\Phi(x, t_n) \in \mathcal{W}_n$, for arbitrary $x \in X$.

Now we make the choice $t = t_n$ and define the sequence $g_n(x) = a(x, t_n)$. The only thing left is to define the homotopies $g_{n,n+1} : X \times I \to M$ by

$$g_{n,n+1}(x,t) = a(x, (1-t)t_n + t t_{n+1}).$$

Lemma 4.2. *The sequence of pairs $(g_n, g_{n,n+1}) : X \to M$ is a strong proximate sequence.* □

The proof of the above lemma is an appropriate adjustment of the proof of a lemma given in the paper [22]. In the following section we will give some insights to the question from [23]: "*Is every shape equivalence $f : M \to X$ a strong shape equivalence?*"

In the special case when f is the inclusion map ($f = i$) and we can define a semi-flow

$$\{\Phi_t : X \to X \mid t \in \mathbb{R}^+\}$$

in X such that $M \subseteq X$ is a global attractor we will make some useful conclusion.

Intrinsic strong shape of attractors.

Theorem 4.3. *Let X be a compact metric space and*

$$\{\Phi_t : X \to X \mid t \in \mathbb{R}^+\}$$

be a semi-dynamical system with a global attractor M. Then the inclusion map $i : M \to X$ induces a strong shape equivalence.

Proof. We define a strong proximate sequence $(f_n, f_{n,n+1}) : M \to X$ by $f_n = i$, for every $n \in \mathbb{N}$ and the homotopies $f_{n,n+1} : M \times I \to X$ are given by $f_{n,n+1}(x,t) = x$, for every $x \in M, t \in I$ and for every $n \in \mathbb{N}$. We will prove that the strong proximate sequence $(g_n, g_{n,n+1})$ is a strong shape inverse of $(f_n, f_{n,n+1})$, i.e., the following holds:

$$[(g_n, g_{n,n+1})] \circ [(f_n, f_{n,n+1})] = [(1_n, 1_{n,n+1})],$$
$$[(f_n, f_{n,n+1})] \circ [(g_n, g_{n,n+1})] = [(1_n, 1_{n,n+1})],$$

where $[(g_n, g_{n,n+1})]$ stands for homotopy class of the strong proximate sequence $(g_n, g_{n,n+1})$ (see the proof of [22, Proposition 3.1] for the equality $[g_n] = [1_n]$ in the shape category).

We will prove the first composition as follows. Let us note that the composition

$$[(g_n, g_{n,n+1})] \circ [(f_n, f_{n,n+1})] = [(h_n, h_{n,n+1})]$$

is given by $h_n = g_n \circ f_{k_n}$, where k_n is an appropriate chosen subsequence for the cofinal sequences of coverings $(\mathcal{V}_n)$ for M and $(\mathcal{U}_n)$ for X according to the definition and the connecting homotopy is given by:

$$h_{n,n+1}(x,t) = \begin{cases} g_n f_{k_n,k_{n+1}}(x, 2t), & t \in [0; \frac{1}{2}], \\ g_{n,n+1}(f_{k_{n+1}}(x), 2t-1), & t \in [\frac{1}{2}; 1]. \end{cases}$$

On the other hand the strong proximate sequence $(1_n, 1_{n,n+1})$ is given by

$$1_n : M \to M, \qquad 1_n(x) = x,$$
$$1_{n,n+1} : M \times I \to M, \qquad 1_{n,n+1}(x,t) = x,$$

for arbitrary $x \in M, t \in I$. We will prove that the strong proximate sequences $(h_n, h_{n,n+1})$ and $(1_n, 1_{n,n+1})$ are homotopic. First let us note that

$$h_n(x) = g_n \circ f_{k_n}(x) = g_n(x) = \Phi(x, t_n).$$

Now define $w_n : M \times I \to M$ by $w_n(x,t) = \Phi(x,(1-t)t_n)$. This is a continuous homotopy connecting h_n and 1_n. We consider the following map $w_{n,n+1} : M \times I \times I \to M$ given by:

$$w_{n,n+1}(x,t,s) = \begin{cases} \Phi(x,(1-s)t_n), & t \in [0;\frac{1}{2}], \\ \Phi(x, 2(1-s)(1-t)t_n + (1-s)(2t-1)t_{n+1}), & t \in [\frac{1}{2};1]. \end{cases}$$

We will prove that $(w_n, w_{n,n+1})$ is a strong proximate sequence connecting $(h_n, h_{n,n+1})$ and $(1_n, 1_{n,n+1})$. Namely

$$\begin{aligned} w_{n,n+1}(x,t,0) &= h_{n,n+1}(x,t) && \text{for } s=0, \\ w_{n,n+1}(x,t,1) &= x = 1_{n,n+1}(x,t) && \text{for } s=1. \end{aligned}$$

Similarly,

$$\begin{aligned} w_{n,n+1}(x,0,s) &= w_n(x,s) && \text{for } t=0, \\ w_{n,n+1}(x,1,s) &= w_{n+1}(x,s) && \text{for } t=1. \end{aligned}$$

Then the claim follows from the continuity of w_n and $w_{n,n+1}$.

It remains to prove the second composition i.e., that

$$[(f_n, f_{n,n+1})] \circ [(g_n, g_{n,n+1})] = [(1_n, 1_{n,n+1})].$$

First let

$$[(f_n, f_{n,n+1})] \circ [(g_n, g_{n,n+1})] = [(h_n, h_{n,n+1})],$$

where the strong proximate sequence $(h_n, h_{n,n+1})$ by definition of composition is given by $h_n = f_n \circ g_{k_n}$ and:

$$h_{n,n+1}(x,t) = \begin{cases} f_n g_{k_n,k_{n+1}}(x,2t), & t \in [0;\frac{1}{2}], \\ f_{n,n+1}(g_{k_{n+1}}(x), 2t-1), & t \in [\frac{1}{2};1]. \end{cases}$$

Let us note that the map $h_n : X \to X$ is given by $h_n(x) = a(x, t_{k_n})$ and using the notation $c = k_{n+1} - k_n$ the homotopy $h_{n,n+1}$ is given by:

$$h_{n,n+1}(x,t) = \begin{cases} a(x,(1-2ct)t_{k_n} + 2ctt_{k_n+1}), & 2t \in [0;\frac{1}{c}], \\ a(x,(2-2ct)t_{k_n+1} + (2ct-1)t_{k_n+2}), & 2t \in [\frac{1}{c};\frac{2}{c}], \\ \cdots\cdots\cdots\cdots & \\ a(x,(c-2ct)t_{k_{n+1}-1} + (2tc-c+1)t_{k_{n+1}}), & 2t \in [\frac{c-1}{c};1], \\ a(x,t_{k_{n+1}}), & t \in [\frac{1}{2};1]. \end{cases}$$

We need to construct a strong proximate sequence $(w_n, w_{n,n+1})$ which connects the strong proximate sequence $(h_n, h_{n,n+1})$ and $(1_n, 1_{n,n+1})$. We proceed as follows. The choice for the homotopy w_n is given by:

$$w_n(x,t)=\begin{cases}\Phi(x,(1-t)t_{k_n}), & t\neq 0,\\ a(x,t_{k_n}), & t=0.\end{cases}$$

It remains to define the homotopy of second order $w_{n,n+1}: X\times I\times I\to X$ with:

$$w_{n,n+1}(x,t,s)=\begin{cases}\Phi(x,((1-p)t_{k_n}+pt_{k_n+1})q), & 2t\in[0;\frac{1}{c}],\\ \Phi(x,((2-p)t_{k_n+1}+(p-1)t_{k_n+2})q), & 2t\in[\frac{1}{c};\frac{2}{c}],\\ \quad\cdots\cdots\cdots\cdots & \\ \Phi(x,((c-p)t_{k_{n+1}-1}+(p-c+1)t_{k_{n+1}})q), & 2t\in[\frac{c-1}{c};1],\\ \Phi(x,t_{k_{n+1}}q), & t\in[\frac{1}{2};1]\end{cases}$$

where $p=2tc$ and $q=1-s\neq 1$. For the case where $q=1-s=1$ we go with:

$$w_{n,n+1}(x,t,0)=\begin{cases}a(x,(1-p)t_{k_n}+pt_{k_n+1}), & 2t\in[0;\frac{1}{c}],\\ a(x,(2-p)t_{k_n+1}+(p-1)t_{k_n+2}), & 2t\in[\frac{1}{c};\frac{2}{c}],\\ \quad\cdots\cdots\cdots\cdots & \\ a(x,(c-p)t_{k_{n+1}-1}+(p-c+1)t_{k_{n+1}}), & 2t\in[\frac{c-1}{c};1],\\ a(x,t_{k_{n+1}}), & t\in[\frac{1}{2};1].\end{cases}$$

We need to check the connecting relations. For the level $s=0$ we have that $w_{n,n+1}(x,t,0)=h_{n,n+1}(x,t)$ and for the level $s=1$ we obtain the relation $w_{n,n+1}(x,t,1)=1_{n,n+1}(x,t)=x$. For the level $t=0$ we have that $w_{n,n+1}(x,0,s)=w_n(x,s)$ and for the level $t=1$ we obtain the relation $w_{n,n+1}(x,1,s)=w_{n+1}(x,s)$.

We will prove that the homotopies $w_n, w_{n,n+1}$ are sufficiently close to continuous according to the definition. Namely let $(h_n,h_{n,n+1})$ and $(1_n,1_{n,n+1})$ be strong proximate sequences over a cofinal sequence of coverings $(\mathcal{U}_n)$ of X with corresponding Lebesgue numbers γ_n such that $\lambda_{k_n}\leqslant\gamma_n(\mathcal{V}_{k_n}\prec\mathcal{U}_n)$. We will prove that $(w_n,w_{n,n+1})$ is a strong proximate sequence over $(\mathrm{st}(\mathcal{U}_n))$ such that:

(i) $w_n: X\times I\to X$ is $\mathcal{U}_n$-homotopy between h_n and 1_n.
(ii) $w_{n,n+1}: X\times I\times I\to X$ is $\mathrm{st}^2(\mathcal{U}_n)$-continuous and at all points from $X\times\partial I^2$ is $\mathrm{st}(\mathcal{U}_n)$-continuous connecting $w_{n,n+1}(x,t,0)=h_{n,n+1}(x,t)$ and $w_{n,n+1}(x,t,1)=1_{n,n+1}(x,t)$.

Let us discuss the first claim. We will prove that the homotopy w_n is $\mathrm{st}(\mathcal{U}_n)$-continuous. Note that for points $(x_0,t_0)\in X\times I\backslash\{0\}$ the homotopy w_n is continuous according to the definition and hence is $\mathrm{st}(\mathcal{U}_n)$-continuous as well. It remains to consider the points $(x_0, 0) \in X \times \{0\}$. The map $h_n : X \to X$ is $\mathcal{U}_n$-continuous, whence there exists a neighborhood W_{x_0} of x_0 such that $h_n(W_{x_0}) \subseteq U_n$, for some $U_n \in \mathcal{U}_n$. Note that

$$\Phi(x_0, t_{k_n}) \in T\big(M, \tfrac{\lambda_{k_n}}{8}\big),$$

so there exists $U_n^* \in \mathcal{U}_n$ such that:

$$\Phi(x_0, t_{k_n}),\ h(x_0) = a(x_0, t_{k_n}) \in U_n^*.$$

We will consider the map $r(x,t) = \Phi(x, (1-t)t_{k_n})$ at the point $(x_0, 0)$. From the continuity of r there exists a neighborhood $U_{x_0} \times Q_0$ of $(x_0, 0)$ such that

$$r(U_{x_0} \times Q_0) \subseteq U_n^*.$$

One can assume that $U_{x_0} \subseteq W_{x_0}$. Now from

$$U_{x_0} \times Q_0 = U_{x_0} \times \{0\} \cup U_{x_0} \times Q_0 \backslash \{0\}$$

we obtain the following inclusion:

$$r(U_{x_0} \times Q_0 \backslash \{0\}) = w_n(U_{x_0} \times Q_0 \backslash \{0\}) \subseteq U_n^*,$$

while from

$$w_n(U_{x_0} \times \{0\}) = h_n(U_{x_0}) \subseteq U_n$$

we get that

$$w_n(U_{x_0} \times Q_0) \subseteq U_n^* \cup U_n.$$

Notice that $U_n^* \cap U_n \neq \varnothing$ hence $w_n(U_{x_0} \times Q_0) \subseteq \mathrm{st}(U_n)$. Emphasize also that for points from $X \times \partial I$ we have that either $w_n(x, 0) = a(x, t_{k_n})$ or $w_n(x, 1) = \Phi(x, 0) = x$. This implies $\mathcal{U}_n$-continuity at those points (recall that $\mathcal{V}_{k_n} \prec \mathcal{U}_n$).

It remains to consider the homotopy of the second order

$$w_{n,n+1} : X \times I \times I \to X$$

and to investigate how close to continuity this map really is. Note that at the points (x_0, t_0, s_0), $s_0 \neq 0$, the homotopy $w_{n,n+1}$ is continuous because of the continuity of the phase map Φ. We will discuss the points $(x_0, t_0, 0)$. The map $h_{n,n+1}$ is $\mathrm{st}(\mathcal{U}_n)$-continuous which means that there exists a neighborhood $U_{x_0} \times T_{t_0}$ of (x_0, t_0) such that:

$$h_{n,n+1}(U_{x_0} \times T_{t_0}) \subseteq \mathrm{st}(U_n), \quad \mathrm{st}(U_n) \in \mathrm{st}(\mathcal{U}_n).$$

It suffices to discuss the case when

$$2t_0 = \tfrac{m}{k_{n+1} - k_n} = \tfrac{m}{c}, \qquad m \in \{1, 2, \ldots, c-1\}.$$

We introduce the point $P = \Phi(x_0, t_{k_n+m})$. Consider the following two continuous maps:

$$r_1(x,t,s) = \Phi\big(x, (m - 2ct)(1-s)t_{k_n+m-1} + (2ct - m + 1)t_{k_n+m}(1-s)\big),$$

$$r_2(x,t,s) = \Phi\big(x, (m+1-2ct)(1-s)t_{k_n+m} + (2ct-m)t_{k_n+m+1}(1-s)\big).$$

Notice that

$$d(P, w_{n,n+1}(x_0,t_0,0)) < \tfrac{\lambda_{k_n+m}}{8} < \gamma_n,$$

whence there exists $U_n^* \in \mathcal{U}_n$ such that

$$P,\ w_{n,n+1}(x_0,t_0,0) \in U_n^*.$$

Now from the continuity of the maps r_1, r_2 there exist neighborhoods

$$U_{x_0}^1 \times T_{t_0}^1 \times W_0^1 \quad \text{and} \quad U_{x_0}^2 \times T_{t_0}^2 \times W_0^2$$

of $(x_0, t_0, 0)$ such that:

$$r_1(U_{x_0}^1 \times T_{t_0}^1 \times W_0^1) \subseteq U_n^*, \qquad r_2(U_{x_0}^2 \times T_{t_0}^2 \times W_0^2) \subseteq U_n^*.$$

We introduce the following sets:

$$W_0^* = W_0^1 \cap W_0^2,\ T_{t_0}^* = T_{t_0} \cap T_{t_0}^1 \cap T_{t_0}^2,\ U_{x_0}^* = U_{x_0} \cap U_{x_0}^1 \cap U_{x_0}^2.$$

Then from the relation:

$$U_{x_0}^* \times T_{t_0}^* \times W_0^* = (U_{x_0}^* \times T_{t_0}^{*+} \times W_0^* \backslash \{0\}) \cup (U_{x_0}^* \times T_{t_0}^{*+} \times \{0\}) \cup \\ \cup (U_{x_0}^* \times T_{t_0}^{*-} \times W_0^* \backslash \{0\}) \cup (U_{x_0}^* \times T_{t_0}^{*-} \times \{0\}),$$

where $T_{t_0}^{*+}$ and $T_{t_0}^{*-}$ stands for:

$$T_{t_0}^{*+} = T_{t_0}^* \cap [t_0, +\infty), \quad \text{i.e.} \quad T_{t_0}^{*-} = T_{t_0}^* \cap (-\infty, t_0].$$

Hence we get the following inclusions:

$$w_{n,n+1}(U_{x_0}^* \times T_{t_0}^{*+} \times W_0^* \backslash \{0\}) = r_2(U_{x_0}^* \times T_{t_0}^{*+} \times W_0^* \backslash \{0\}) \subseteq U_n^* \subseteq \mathrm{st}(U_n^*),$$

$$w_{n,n+1}(U_{x_0}^* \times T_{t_0}^{*+} \times \{0\}) = h_{n,n+1}(U_{x_0}^* \times T_{t_0}^{*+}) \subseteq \mathrm{st}(U_n),$$

as well as

$$w_{n,n+1}(U_{x_0}^* \times T_{t_0}^{*-} \times W_0^* \backslash \{0\}) = r_1(U_{x_0}^* \times T_{t_0}^{*-} \times W_0^* \backslash \{0\}) \subseteq U_n^* \subseteq \mathrm{st}(U_n^*),$$

$$w_{n,n+1}(U_{x_0}^* \times T_{t_0}^{*-} \times \{0\}) = h_{n,n+1}(U_{x_0}^* \times T_{t_0}^{*-}) \subseteq \mathrm{st}(U_n).$$

Note that $\mathrm{st}(U_n^*) \cap \mathrm{st}(U_n) \neq \varnothing$ which implies that

$$w_{n,n+1}(U_{x_0}^* \times T_{t_0}^* \times W_0^*) \subseteq \mathrm{st}^2(U_n).$$

Hence we conclude $\mathrm{st}^2(\mathcal{U}_n)$-continuity.

It remains to discuss the continuity at the points from $X \times \partial I^2$. For the level $s = 0$ we have $w_{n,n+1}(x,t,0) = h_{n,n+1}(x,t)$ and for $s = 1$ we obtain the relation $w_{n,n+1}(x,t,1) = 1_{n,n+1}(x,t) = x$. For the level $t = 0$ we have $w_{n,n+1}(x,0,s) = w_n(x,s)$ and for $t = 1$ we obtain the relation $w_{n,n+1}(x,1,s) = w_{n+1}(x,s)$. Hence we conclude $\mathrm{st}(\mathcal{U}_n)$-continuity. □

Now we shall use the following theorem from [4]

Theorem 4.4. *Let $f : X \to Y$ be a map. The following holds:*

1) *The map $f : X \to Y$ is a shape equivalence if and only if the inclusion $i : X \to M(f)$ from X to the mapping cylinder $M(f)$ of f is a shape equivalence.*
2) *The map $f : X \to Y$ is a strong shape equivalence if and only if the inclusion $i : X \to M(f)$ from X to the mapping cylinder $M(f)$ of f is a strong shape equivalence.*

The previous two Theorems 4.3 and 4.4 and discussion gives us a new insight of the already mentioned problem from [23]. Namely, now we can state the following question which arises from the results above and is purely dynamic by nature:

Problem 4.5. Given a set X in a space $M(f)$ such that the inclusion $i : X \to M(f)$ induces a shape equivalence, when is X the global attractor for some semi-dynamical system in $M(f)$?

By answering this question we obtain a substantial information about the problem in [23]. For example note that if X has the homotopy type of a compact ANR then it is well known that X can be realized as a global attractor in $M(f)$ for a suitably defined semi-flow. Hence we have the following claim:

Corollary 4.6. *Let $f : X \to Y$ be a shape equivalence. If X has the homotopy type of a compact ANR then f is a strong shape equivalence as well.*

This means that we can use dynamical tools to answer shape theoretical problems as well. Attacking Problem 4.5 which is a dynamical problem by nature we actually obtain, at least partially, an answer to the purely shape theoretical problem from [23].

Note that our Theorem 4.3 does not hold for discrete dynamical systems as the following example shows.

Example 4.7. Consider the discrete semi-group $\{\Phi_n\}$ on $\mathbb{R}^2$ generated by the function $f(x, y) = (\cos 10x, \cos 10y)$, for every $(x, y) \in \mathbb{R}^2$. Let us introduce the following set $A = \{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$. Since $f(\mathbb{R}^2) = A$ and $f(A) = A$, it is clear that A is the global attractor for $\{\Phi_n\}$. On hte other hand, the strong shape of the plane clearly differs from the circle A, being contractible.

We will mention the following interesting corollary of our result as well which suggest a kind of topological robustness of global attractors in nice spaces.

Corollary 4.8. *Let $\Phi_\lambda : X \times \mathbb{R} \to X, \lambda \in I$ be a parameterized family of flows defined on a locally compact ANR, X. If M is an attractor of Φ_0 then for every neighborhood V of M contained in the basin of attraction of M there exists a λ_0, with $0 < \lambda_0 \leqslant 1$, such that for every $\lambda \leqslant \lambda_0$ there exists an attractor $M_\lambda \subset V$ of the flow Φ_λ with $SSh(M_\lambda) = SSh(M)$. Moreover V is contained in the basin of attraction of M_λ.*

Connectedness properties of attractors in metric spaces. The following theorem is a nice build up of our previous theorem. Namely, using Theorem 4.3 we will give a nice result which also improves some of the results from [10].

Theorem 4.9. *Let X be a compact metric space and let*

$$\{\Phi_t : X \to X \mid t \in \mathbb{R}^+\}$$

be a semi-flow with a global attractor M. Then there exists a bijection $\varphi : \square(M) \to \square(X)$ between the spaces of components of the global attractor M and X such that $M_0 \subseteq \varphi(M_0)$ for every $M_0 \in \square(M)$ and such that the inclusion $i : M_0 \to \varphi(M_0)$ induces a strong shape equivalence.

Proof. For every component $X_\alpha \in \square(X)$ put $M_\alpha = X_\alpha \cap M$. Since the semi-trajectory $\gamma^+(x)$ of every point $x \in X_\alpha$ is connected and M is compact, we have that $\varnothing \neq \omega(x) \subseteq M_\alpha$. We shall prove that M_α is connected.

Suppose that M_α is disconnected, so $M_\alpha = M_\alpha^1 \cup M_\alpha^2$, where M_α^1 and M_α^2 are nonempty disjoint compacta. Consider the following sets

$$X_\alpha^i = \{x \in X_\alpha \mid \omega(x) \subseteq M_\alpha^i\}, i = 1, 2.$$

Note that X_α^1 and X_α^2 are disjoint by definition and since $M_\alpha^i \subseteq X_\alpha^i$, for $i = 1, 2$ they are nonempty. We shall prove that $X_\alpha = X_\alpha^1 \cup X_\alpha^2$.

Let $x \in X_\alpha$. By the normality of X there exist neighborhoods U_1 and U_2 of M_α^1 and M_α^2 correspondingly such that $U_1 \cap U_2 = \varnothing$. Hence there exists $T > 0$ such that

$$\gamma^+(\Phi(x, T)) \subseteq U_1 \cup U_2.$$

Since $\gamma^+(\Phi(x, T))$ is connected, we have that either $\gamma^+(\Phi(x, T)) \subseteq U_1$ or $\gamma^+(\Phi(x, T)) \subseteq U_2$. Therefore $\omega(x) \subseteq M_\alpha^1$ or $\omega(x) \subseteq M_\alpha^2$. Finally, we shall prove that X_α^1 and X_α^2 are closed sets.

Let us choose an arbitrary sequence $x_n \in X_\alpha^1$ converging to p, i.e., $x_n \to p$. Suppose that $p \notin X_\alpha^1$. Then $p \in X_\alpha^2$. Hence there exists $T_p \in \mathbb{R}$ such that $\Phi(p, t) \in U_2$ for every $t \geqslant T_p$. On the other hand using a theorem from [1] for $K = X_\alpha \backslash (U_1 \cup U_2) \neq \varnothing$ (in the opposite case X_α is disconnected and the proof is complete) there exists a Lyapunov function $L : X \to \mathbb{R}^+$ such that for sufficiently small $\beta > 0$, $L(x) \geqslant \beta$, for every $x \in K$. Note that M_α^2 attracts $\{p\}$ hence there exists $t_p \in \mathbb{R}, t_p \geqslant T_p$ such that $L(\Phi(p, t_p)) < \frac{\beta}{2}$.

From the continuity of L and Φ there exists a neighborhood V of p such that

$$L(\Phi(x, t_p)) < \frac{\beta}{2} \text{ for all } x \in V.$$

Also from the continuity of Φ there exists a neighborhood W of p such that $\Phi(x, t_p) \in U_2$ for every $x \in W$. Now for sufficiently large n, $x_n \in V \cap W$, whence

$$L(\Phi(x_n, t_p) < \frac{\beta}{2} \quad \text{and} \quad \Phi(x_n, t_p) \in U_2.$$

On the other hand from the fact that $x_n \in X^1_\alpha$ there exists $t \geqslant t_p$ such that $\Phi(x_n, t) \in U_1$. Consider the connected set $Q = \Phi(x_n, [t_p, \infty))$. If $z \in K \cap Q \neq \emptyset$ then from the property of the Lyapunov function we have that $L(z) < \frac{\beta}{2}$ but from the theorem from [1] we have that $L(z) \geqslant \beta$, a contradiction. So the only possibility left is $Q \subset U_1 \cup U_2$. However since

$$Q \cap U_1 \neq \emptyset \quad \text{and} \quad Q \cap U_2 \neq \emptyset,$$

we obtain that Q is disconnected, which gives a contradiction.

Thus we conclude that X^1_α is closed. Similarly, X^2_α is closed. Hence X_α is not connected which contradicts to the assumption. Therefore, we have proved that M_α is connected. Finally, we define $\varphi : \square(M) \to \square(X)$ by $\varphi(M_\alpha) = X_\alpha$. Then the last conclusion follows from the previous Theorem 4.3. □

Remark 4.10. Theorem 4.9 holds for arbitrary metric spaces. The compactness condition is required for the last conclusion that the inclusion $i : M_0 \to \varphi(M_0)$ induces a strong shape equivalence.

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