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INTERPOLATION POLYNOMIALS VIA AIFS

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Abstract. The general theory of IFS - AIFS relations is applied to the case of polynomial attractors. The starting point is Bézier subdivision embodied in two linear contractions of $\mathbb{R}^n$ space.

1. Introduction

The concept of Iterated Function System (IFS), and its affine invariant counterpart AIFS appear to play a crucial role in constructive theory of fractal sets and in paving the way to have a good modeling tools for such sets. But, if the collection of objects to be modeled, besides fractals contains smooth objects as well (polynomials for ex.) then one needs to revisit classical algorithms for smooth objects generation and to introduce the new one that is capable to create both fractal and smooth forms. In this light, the purpose of this paper is to develop such algorithms for interpolating polynomials.

Let $\{w_i,\, i = 1, 2, ..., n\}$, $n > 1$, be a set of contractive affine mappings defined on the complete Euclidian metric space $(\mathbb{R}^m, d_E)$

$$w_i(\mathbf{x}) = \mathbf{A}_i\mathbf{x} + \mathbf{b}_i, \quad \mathbf{x} \in \mathbb{R}^m, \quad i = 1, 2, ..., n, \tag{1.1}$$

where $\mathbf{A_i}$ is an $m\times m$ real matrix and $\mathbf{b_i}$ is an m-dimensional real vector. Supposing that the Lipschitz factors $s_i = \mathrm{Lip}\{w_i\}$, satisfy condition $|s_i| < 1,\ \ i = 1, 2, ..., n$, the system $\{\mathbb{R}^m; w_1, w_2, ..., w_n\}$ is called (hyperbolic) Iterated Function System (IFS). Associated with given IFS, so called *Hutchinson operator* $W : \mathcal{H}(\mathbb{R}^m) \to \mathcal{H}(\mathbb{R}^m)$, defined by

$$W(B) = \bigcup_{i=1}^{n} w_i(B)\,,\ \forall B \in \mathcal{H}(\mathbb{R}^m) \tag{1.2}$$

is a contractive mapping on the complete metric space $(\mathcal{H}(\mathbb{R}^m), h)$ with contractivity factor $s = \max\{s_i\}$. Here, is the space of nonempty compact subsets of $\mathbb{R}^m$ and h stands for Hausdorff metric induced by d_E, i.e.

$$h(A, B) = \max\left\{\max_{a\in A}\ \min_{b\in B} d_E(a, b), \max_{b\in B}\ \min_{a\in A} d_E(b, a)\right\},\ \forall A, B \in \mathcal{H}(\mathbb{R}^m)\,.$$

According to the contraction mapping theorem, W has the unique fixed point, $A \in \mathcal{H}(\mathbb{R}^m)$ called the *attractor* of the IFS, satisfying.

$$A = W(A) = \bigcup_{i=1}^{n} w_i(A)$$

Definition 1.1. A (non-degenerate) *m-dimensional simplex* $\hat{\mathbf{P}}_{\mathbf{m}}$ (or *simplex*) is the convex hull of a set of $m+1$ affinely independent points (vectors) $\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_{m+1}$ in Euclidean space of dimension m or higher, $\hat{\mathbf{P}}_m = \operatorname{conv}\{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_{m+1}\}$. The vertices of $\hat{\mathbf{P}}_m$ will be denoted by $\mathbf{P}_m$ and represented by the vector

$$\mathbf{P}_m = \begin{bmatrix} \mathbf{p}_1^{\mathrm{T}} & \mathbf{p}_2^{\mathrm{T}} & \ldots & \mathbf{p}_{m+1}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}.$$

Let $\mathrm{S}_{m+1} = [s_{i,j}]_{i,j=1}^{m+1}$ be an $(m+1) \times (m+1)$ row-stochastic real matrix (its rows sum up to 1).

Definition 1.2. We refer to the linear mapping $\mathcal{L}\colon \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$, such that $\mathcal{L}(\mathbf{x})=\mathrm{S}^{\mathrm{T}}\mathbf{x}$ as the *linear mapping associated with* S.
The corresponding Hutchinson operator is

$$W'(B) = \bigcup_{i=1}^{n} \mathcal{L}_i(B), \quad \forall B \in \mathcal{H}\left(\mathbb{R}^{m+1}\right) \tag{1.3}$$

Definition 1.3. Let $\hat{\mathbf{P}}_m$ be a non-degenerate simplex and let $\{\mathrm{S}_i\}_{i=1}^{n}$ be a set of real square nonsingular row-stochastic matrices of order $m+1$. The system $\Omega\left(\hat{\mathbf{P}}_m\right) = \left\{\hat{\mathbf{P}}_m; \mathrm{S}_1, \mathrm{S}_2, ..., \mathrm{S}_n\right\}$ is called *(hyperbolic) Affine invariant IFS (AIFS)*, provided that the linear mappings associated with S_i are contractions in $(\mathbb{R}^m, d_E)$ ([4 − 6]).

Theorem 1. *One eigenvalue of the matrix* S_i *is 1, other m eigenvalues coincide with eigenvalues of* $\mathbf{A}_i$*, the matrix that makes the linear part of the affine mapping* w_i *given by* (1.1)*. In other words,* $\operatorname{sp}\{\mathrm{S}_i\} = \operatorname{sp}\{\mathbf{A}_i\} \cup \{1\}$.

2. Polynomials and subdivision

Although the notion of subdivision is usually attributed to m-dimensional $(m \geq 1)$ continuous parametric mapping $t \mapsto P_n(t),\ t \in [a,b] \quad (a < b)$, so that $P_n(t) \in \mathbb{R}^m$, it is enough to consider an one-dimensional case, i.e., $m = 1$. Also, according to [3] the restriction on $P_n(t) \in \mathbb{R}$ to be an algebraic polynomial allows considering only linear subdivision. More precise, if

$$P_n(t) = \sum_{k=0}^{n} A_k\, B_k^n(t), \quad t \in [a,b] \tag{2.1}$$

where A_k are real coefficients and

$$\mathcal{B}_n(t) = \{B_0^n(t),\ B_1^n(t), ..., B_n^n(t)\}, \quad t \in [a,b], \tag{2.2}$$

is some functional basis, it may happened that both A_k and $\mathcal{B}_n(t)$ depend on the interval of definition. To stress this fact, in the case when confusion may appear, it is suitable to write $A_k[a,b]$ as well as $\mathcal{B}_n[a,b](t)$. Then, the subdivision is defined as follows.

Definition 2.1. The function P_n, defined by (2.1) is said to *permit linear subdivision* if and only if for each nonempty subinterval $[p,q] \subset [a,b]$, there exists a set of coefficients $\{A_k\,[p,q]\}_{k=0}^{n}$ such that

$$\sum_{k=0}^{n} A_k\,[p,q]\,B_k^n\,[p,q](t) = \sum_{k=0}^{n} A_k\,[a,b]\,B_k^n\,[a,b]\,(\varphi(t))$$

for $t \in [a,b]$, where

$$\varphi(t) = \frac{1}{b-a}\,((q-p)t + b\,p - a\,q)$$

is the affine contraction that maps $[a,b]$ into $[p,q]$.

As it is shown by Goldman and Heath, linear subdivision is strictly a polynomial phenomenon.

Theorem 2. ([3]) *The function P_n, defined by* (2.1) *admit linear subdivision if and only if $\mathcal{B}_n(t)$ is a polynomial basis.*

The classic, and best known subdivision phenomena is connected with Bernstein polynomial basis $\{b_k^n\}_{k=0}^{n}$, of degree $n \in \mathbb{N}^0$, where the basis functions $t \mapsto b_k^n(t)$ are usually defined on the interval $[0,1]$

$$b_k^n(x) = \left(\begin{array}{c} n \\ k \end{array}\right) x^k (1-x)^{n-k}, \quad k = 0, ..., n$$

or, on $[a,b]$

$$b_k^n[a,b](x) = \frac{1}{(b-a)^n}\left(\begin{array}{c} n \\ k \end{array}\right)(x-a)^k\,(b-x)^{n-k}, \quad k = 0, ..., n.$$

It is suitable, for the sake of operational efficacy, to write Bernstein basis in vector form

$$\mathbf{b}_{[a,\,b]}(x) = \Big[b_0^n[a,\,b](x) \quad b_1^n[a,\,b](x) \ \ \dots \ \ b_n^n[a,\,b](x)\Big]^{\mathrm{T}}.$$

Also, the convention $\mathbf{b}_{[0,1]}(x) = \mathbf{b}(x)$ will be adopted.

Theorem 3. *(Subdivision) For all $0 \le p < q \le 1$, it holds*

$$b_k^n\,[(1-t)p + tq] = \sum_{m=0}^{n}\left(\sum_{i+j=k} b_i^{n-m}(p)\,b_j^m(q)\right) b_m^n(t), \quad t \in [0,1].$$

The span of Bernstein basis $\mathbf{b}_{[a,b]}\ (x)$ of degree n, with coefficient vector $\mathbf{p}_{[a,b]} = \left[\begin{array}{cccc} p_0^{[a,b]} & p_1^{[a,b]} & \dots & p_n^{[a,b]} \end{array}\right]^{\mathrm{T}}$, is known as the *n-th degree Bernstein polynomial* $B_n(\mathbf{p}_{[a,b]}x) = \mathbf{p}_{[a,b]}^{\mathrm{T}}\,\mathbf{b}_{[a,b]}(x)$. The indices $[a,b]$ in sub-or superscripts of the coefficient vector indicate that these coefficient depend on the interval of definition of Bernstein polynomial.

Let $[c,d]$ be an arbitrary subinterval of $[a,b]$ such that $c=(1-p)\,a+pb$, $d=(1-q)\,a+qb$. Then, the direct consequence of Theorem 3 is that the Bernstein polynomial over $[c,d]$ is given by another coefficient vector, $\mathbf{p}_{[c,d]}$ i.e., $B_n(\mathbf{p}_{[c,d]};x)=\mathbf{p}_{[c,d]}^{\mathrm{T}}\,\mathbf{b}_{[c,d]}(x)$. This new coefficient vector is given by [9]

$$\mathbf{p}_{[c,d]}=\begin{bmatrix} p_0^{[c,d]} & p_1^{[c,d]} & \dots & p_n^{[c,d]} \end{bmatrix}^{\mathrm{T}},$$

where $p_m^{[c,d]}=\sum\limits_{k=0}^{n}\left(\sum\limits_{i+j=k} b_i^{n-m}(p)b_j^m(q)\right)p_k$, or, in matrix form

$$\mathbf{p}_{[c,d]}=S_{p,q}\cdot\mathbf{p}_{[a,b]},$$

where

$$S_{p,q}=[s_{m,k}]_{\substack{m=0,n\\k=0,n}}=\left[\sum_{i+j=k} b_i^{n-m}(p)b_j^m(q)\right]_{\substack{m=0,n\\k=0,n}}. \tag{2.3}$$

The matrix $\mathrm{S}_{p,q}$, is called *subdivision matrix*. Its order exceeds degree of $B_n(\mathbf{p}_{[a,b]};x)$ for one. For example, a subdivision matrix for $n=3$, reads

$$S_{p,\,q}^{\mathrm{T}}=\begin{bmatrix} (1-p)^3 & 3(1-p)^2p & 3(1-p)p^2 & p^3 \\ (1-p)^2(1-q) & 2(1-p)p(1-q)+(1-p)^2q & p^2(1-q)+2(1-p)pq & p^2q \\ (1-p)(1-q)^2 & p(1-q)^2+2(1-p)(1-q)q & 2p(1-q)q+(1-p)q^2 & pq^2 \\ (1-q)^3 & 3(1-q)^2q & 3(1-q)q^2 & q^3 \end{bmatrix}.$$

Note that, for "reconstruction" of the whole segment of the polynomial $B_n(\mathbf{p}_{[a,b]};\,x)$ over the interval $[a,b]$, the following three subdivision matrices are necessary: $S_{0,p}, S_{p,q}$, and $S_{q,1}$. These matrices define three new coefficient vectors, $\mathbf{p}_{[a,c]}$, $\mathbf{p}_{[c,d]}$ and $\mathbf{p}_{[d,b]}$ that further yield three sub-segments : $B_n(\mathbf{p}_{[a,c]};\;x)$, $x\in[a,c]$, $B_n(\mathbf{p}_{[c,d]};\;x)$, $x\in[c,d]$, and $B_n(\mathbf{p}_{[d,b]};\,x)$, $x\in[d,b]$ of the starting polynomial. This is the reason for the name *ternary subdivision*, for this process. The simpler, *binary subdivision*, occurs when $[a,b]$ splits into two subintervals, $[a,c]$ and $[c,b]$, $c=(1-\lambda)\,a+\lambda b$. In this case, only two subdivision matrices exist: the "left" one $S_{0,\lambda}$ that will be denoted by $S_{\mathrm{L}}(\lambda)$, and the "right" one $S_{\mathrm{R}}(\lambda)=S_{\lambda,1}$. It follows from (2.3) that

$$S_{\mathrm{L}}(\lambda)=\begin{bmatrix} b_0^0(\lambda) & 0 & 0 & \cdots & 0 \\ b_0^1(\lambda) & b_1^1(\lambda) & 0 & & 0 \\ b_0^2(\lambda) & b_1^2(\lambda) & b_2^2(\lambda) & & 0 \\ \vdots & & & \ddots & \\ b_0^n(\lambda) & b_1^n(\lambda) & b_2^n(\lambda) & & b_n^n(\lambda) \end{bmatrix},$$

$$S_{\mathrm{R}}(\lambda)=\begin{bmatrix} b_0^n(\lambda) & b_1^n(\lambda) & b_2^n(\lambda) & \cdots & b_n^n(\lambda) \\ 0 & b_0^{n-1}(\lambda) & b_1^{n-1}(\lambda) & & b_{n-1}^{n-1}(\lambda) \\ 0 & 0 & b_0^{n-2}(\lambda) & & b_{n-2}^{n-2}(\lambda) \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & & b_0^0(\lambda) \end{bmatrix}.$$

Now, the relations between the new and old coefficient vectors are

$$\mathbf{p}_{[c,b]} = S_{\mathrm{R}}(\lambda)\cdot\mathbf{p}_{[a,b]} \tag{2.4}$$

Also, $S_{\mathrm{R}}(\lambda)=\Pi\cdot S_{\mathrm{L}}(1-\lambda)\cdot\Pi$, where Π is a permutation $(n+1)$-matrix having unit counter-diagonal. Since $\Pi^2=\mathrm{I}$, it holds that $S_{\mathrm{L}}(\lambda)=\Pi\cdot S_{\mathrm{R}}(1-\lambda)\cdot\Pi$ as well.

Example 1. Let the Bernstein polynomial $B_4(\mathbf{p}_{[a,b]};\,x)=\mathbf{p}_{[a,b]}^{\mathrm{T}}\,\mathbf{b}_{[a,b]}(x)$ be defined by the coefficient vector $\mathbf{p}_{[a,b]}=\begin{bmatrix}-1 & -1.5 & -3 & 2\end{bmatrix}^{\mathrm{T}}$, over the interval $[a,b]=[-1\ \ 2.5]$. The coefficient vector represents the ordinates of so called Bézier points whose abscise are $\{a+k(b-a)/n, k=0,...,n\}$. In our case, *Bézier points* are

$$\mathbf{P}=\{(-1,\ \ -1)\,,(-0.125,\ \ 1)\,,(0.75,\ \ -1.5)\,,(1.625,\ \ -3)\,,(2.5,\ \ 2)\}$$

and they are connected by a polygonal line in Fig. 1(a). The graph of the polynomial itself interpolates the end Bézier points. The choice $c=1.1\in[a,b]$ gives the subdivision factor $\lambda=0.6$, and the subdivision matrices are

$$S_{\mathrm{L}}(\lambda)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-\lambda & \lambda & 0 & 0 & 0 \\ (1-\lambda)^2 & 2(1-\lambda)\lambda & \lambda^2 & 0 & 0 \\ (1-\lambda)^3 & 3(1-\lambda)^2\lambda & 3(1-\lambda)\lambda^2 & \lambda^3 & 0 \\ (1-\lambda)^4 & 4(1-\lambda)^3\lambda & 6(1-\lambda)^2\lambda^2 & 4(1-\lambda)\lambda^3 & \lambda^4 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 & 0 \\ 0.16 & 0.48 & 0.36 & 0 & 0 \\ 0.064 & 0.288 & 0.432 & 0.216 & 0 \\ 0.0256 & 0.1536 & 0.3456 & 0.3456 & 0.1296 \end{bmatrix} \quad \text{and}$$

$$S_{\mathrm{R}}(\lambda)=\begin{bmatrix} (1-\lambda)^4 & 4(1-\lambda)^3\lambda & 6(1-\lambda)^2\lambda^2 & 4(1-\lambda)\lambda^3 & \lambda^4 \\ 0 & (1-\lambda)^3 & 3(1-\lambda)^2\lambda & 3(1-\lambda)\lambda^2 & \lambda^3 \\ 0 & 0 & (1-\lambda)^2 & 2(1-\lambda)\lambda & \lambda^2 \\ 0 & 0 & 0 & 1-\lambda & \lambda \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0256 & 0.1536 & 0.3456 & 0.3456 & 01296 \\ 0 & 0.064 & 0.288 & 0.432 & 0.216 \\ 0 & 0 & 0.16 & 0.48 & 0.36 \\ 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

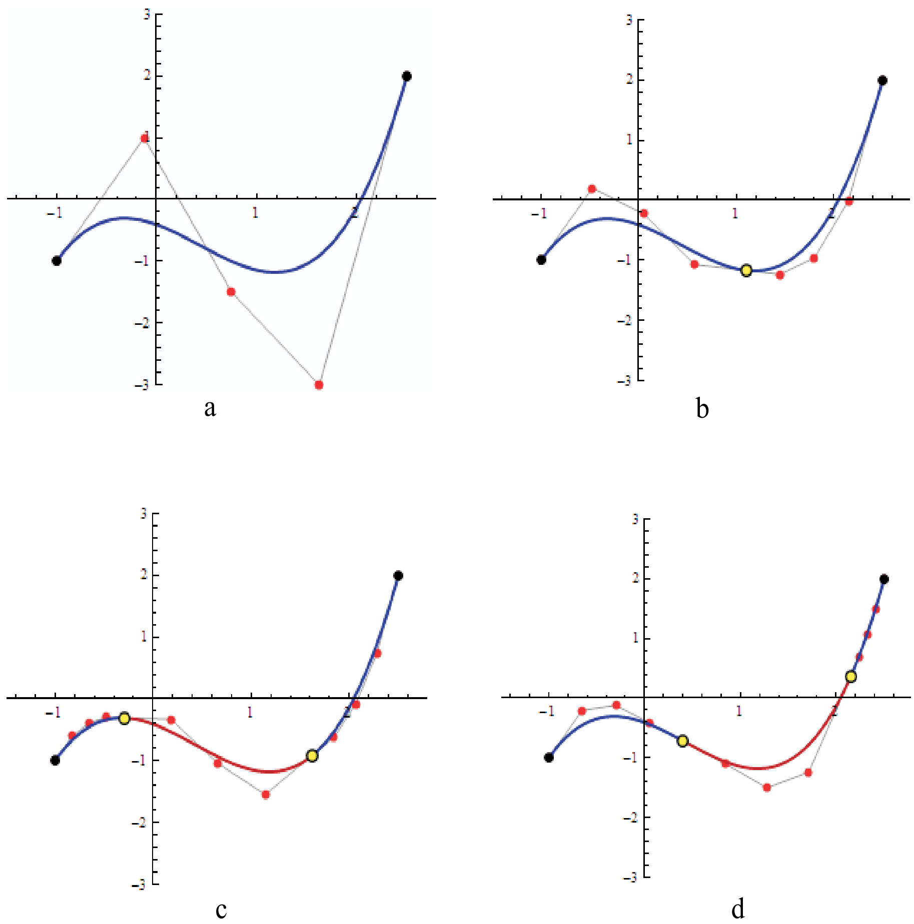

FIGURE 1. Binary and ternary subdivision of Bernstein polynomial

The new Bézier points

$$\mathbf{P}_{\mathrm{L}} = \{(-1,\ \ -1)\,, (0.475,\ \ 0.2)\,, (0.05,\ \ 0.22)\,, (0.575,\ \ 1.072)\,, (1.1,\ \ -1.168)\}$$

and

$$\mathbf{P}_{\mathrm{R}} = \{(1.1,\ \ -1.168)\,, (1.45,\ \ -1.232)\,, (1.8,\ \ -0.96)\,, (2.15,\ \ 0)\,, (2.5,\ \ 2)\}$$

are shown in Fig. 1(b), where the subdivision point is visible. The Fig. 1(c) and Fig. 1(d) represents ternary subdivision for $(p_1, q_1) = (0.2, 0.75)$ and $(p_2, q_2) = (0.4, 0.9)$ respectively. The corresponding subdivision matrices are

$$S_{p_1,q_1} = \begin{bmatrix} 0.4096 & 0.4096 & 0.1536 & 0.0256 & 0.0016 \\ 0.1280 & 0.4800 & 0.3120 & 0.0740 & 0.0060 \\ 0.0400 & 0.2600 & 0.4825 & 0.1950 & 0.0225 \\ 0.0125 & 0.1156 & 0.3656 & 0.4219 & 0.0844 \\ 0.0039 & 0.0469 & 0.2109 & 0.4219 & 0.3164 \end{bmatrix},$$

$$S_{p_2,q_2} = \begin{bmatrix} 0.1296 & 0.3456 & 0.3456 & 0.1536 & 0.0256 \\ 0.0216 & 0.2376 & 0.4176 & 0.2656 & 0.0576 \\ 0.0036 & 0.0696 & 0.3796 & 0.4176 & 0.1296 \\ 0.0006 & 0.0166 & 0.1566 & 0.5346 & 0.2916 \\ 0.0001 & 0.0036 & 0.0486 & 0.2916 & 0.6561 \end{bmatrix}.$$

As soon as the subdivision matrices are determined, a hyperbolic AIFS is defined. The simplest AIFS is the binary one $\Omega(\mathbf{P}) = \{\mathbf{P};\ S_{\mathrm{L}}(\lambda),\ S_{\mathrm{R}}(\lambda)\}$. The hyperbolic character of Ω is the consequence of Theorem 1, i.e., of small sprectral radii $\rho(S_{\mathrm{L}}(\lambda)) = \lambda$ and $\rho(S_{\mathrm{R}}(\lambda)) = 1-\lambda$, both being smaller than 1. The known theorem from matrix theory says that for any square matrix $\mathbf{A}$, the norm $\|\,\boldsymbol{.}\,\|$ can be chosen so to make $\|\mathbf{A}\|$ arbitrarily close to $\rho(\mathbf{A})$. This means that some algorithms for generating fractal sets ([2]) can be applied for generating graphs of smooth polynomials.

3. Monomial basis

Consider the following problem. Let the polynomial

$$P_n(x) = \bar{a}_0 + \bar{a}_1 x + \ldots + \bar{a}_{n-1}x^{n-1} + \bar{a}_n x^n, \quad x \in [a,b] \tag{3.1}$$

be given, by the vector of its real coefficients $\bar{\mathbf{a}} = [\bar{a}_0 \quad \bar{a}_1 \quad \ldots \quad \bar{a}_{n-1} \quad \bar{a}_n\ ]^{\mathrm{T}}$. With the monomial basis on $[a,b]$ that takes vector form,

$$\mathbf{m}(x) = \begin{bmatrix} 1 & \dfrac{x-a}{b-a} & \left(\dfrac{x-a}{b-a}\right)^2 & \cdots & \left(\dfrac{x-a}{b-a}\right)^n \end{bmatrix}^T$$

polynomial (3.1) can be written shortly as

$$P_n(x) = \mathbf{a}^{\mathrm{T}} \cdot \mathbf{m}(x), \qquad x \in [a,b], \tag{3.2}$$

where

$$\mathbf{a} = \begin{bmatrix} a_0 & a_1 & \ldots & a_{n-1} & a_n \end{bmatrix}^{\mathrm{T}}$$

Let the interval $[a,b]$ be split in two subintervals by the point $c \in (a,b)$. Find the "left" and "right" coefficients vectors, $\mathbf{a}^{\mathrm{L}} = [a_0^{\mathrm{L}} \quad a_1^{\mathrm{L}} \quad \ldots \quad a_{n-1}^{\mathrm{L}} \quad a_n^{\mathrm{L}}\ ]^{\mathrm{T}}$ and $\mathbf{a}^{\mathrm{R}} = [a_0^{\mathrm{R}} \quad a_1^{\mathrm{R}} \quad \ldots \quad a_{n-1}^{\mathrm{R}} \quad a_n^{\mathrm{R}}\ ]^{\mathrm{T}}$, such that the "left" polynomial

$$P_n^L(x) = (\mathbf{a}_{\mathrm{L}})^{\mathrm{T}} \cdot \mathbf{m}_{\mathrm{L}}(x), \quad x \in [a,c]$$

where

$$\mathbf{m}_{\mathrm{L}}(x) = \begin{bmatrix} 1 & \dfrac{x-a}{c-a} & \left(\dfrac{x-a}{c-a}\right)^2 & \cdots & \left(\dfrac{x-a}{c-a}\right)^n \end{bmatrix}^{\mathrm{T}}$$

is the "left" monomial basis, coincides with $P_n(x)$ on the left subinterval $[a, c]$, and, at the same time, the "right" polynomial

$$P_n^{\mathrm{R}}(x) = (\mathbf{a}_{\mathrm{R}})^{\mathrm{T}} \cdot \mathbf{m}_{\mathrm{R}}(x), \qquad x \in [c, b]$$

where

$$\mathbf{m}_{\mathrm{R}}(x) = \begin{bmatrix} 1 & \dfrac{x-c}{b-c} & \left(\dfrac{x-c}{b-c}\right)^2 & \cdots & \left(\dfrac{x-c}{b-c}\right)^n \end{bmatrix}^{\mathrm{T}}$$

coincides with $P_n(x)$ on the right subinterval $[c, b]$. In fact, it is a binary subdivision problem for the monomial basis. Now, the monomial and Bernstein basis of degree n are connected by

$$\mathbf{m}(x) = \mathbf{T}_{\mathrm{BM}} \cdot \mathbf{b}(x) \tag{3.3}$$

where $\mathbf{T}_{\mathrm{BM}}$ is an $n \times n$ upper-triangular matrix of the form (for $n = 1, 2, 3$ and 4)

$$[1]\,, \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1/4 & 1/2 & 3/4 & 1 \\ 0 & 0 & 1/6 & 1/2 & 1 \\ 0 & 0 & 0 & 1/4 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ \ldots$$

Multiplying both sides of (3.3) by the vector $\mathbf{a}^{\mathrm{T}}$ from the left yields the last expression is Bernstein polynomial, defined on $[0, 1]$. Now, subdivision for given λ (formulae (2.4)) results in $\mathbf{p}_{\mathrm{L}} = S_{\mathrm{L}}(\lambda) \cdot \mathbf{p}$ and $\mathbf{p}_{\mathrm{R}} = S_{\mathrm{R}}(\lambda) \cdot \mathbf{p}$. On the other hand, by inverting (3.3), which is always possible since all matrices $\mathbf{T}_{\mathrm{BM}}$ are regular, so that $(\mathbf{T}_{\mathrm{BM}})^{-1}$ always exists, one gets $\mathbf{b}(x) = \mathbf{T}_{\mathrm{MB}} \cdot \mathbf{m}(x)$, where it is set $(\mathbf{T}_{\mathrm{BM}})^{-1} = \mathbf{T}_{\mathrm{MB}}$. Consequently,

$$\mathbf{p}_{\mathrm{L}}^{\mathrm{T}} \cdot \mathbf{b}(x) = \left(\mathbf{p}_{\mathrm{L}}^{\mathrm{T}} \cdot \mathbf{T}_{\mathrm{MB}}\right) \cdot \mathbf{m}(x) = \left(\mathbf{T}_{\mathrm{MB}}^{\mathrm{T}} \cdot \mathbf{p}_{\mathrm{L}}\right)^{\mathrm{T}} \cdot \mathbf{m}(x),$$

which results in $\mathbf{a}_{\mathrm{L}} = \mathbf{T}_{\mathrm{MB}}^{\mathrm{T}} \cdot \mathbf{p}_{\mathrm{L}}$. Similarly, $\mathbf{a}_{\mathrm{R}} = \mathbf{T}_{\mathrm{MB}}^{\mathrm{T}} \cdot \mathbf{p}_{\mathrm{R}}$.

Now, taking into account that $\mathbf{p}_{\mathrm{L}} = S_{\mathrm{L}}(\lambda) \cdot \mathbf{p}$ and $\mathbf{p}_{\mathrm{R}} = S_{\mathrm{R}}(\lambda) \cdot \mathbf{p}$, and that $\mathbf{p} = \mathbf{T}_{\mathrm{BM}}^{\mathrm{T}} \cdot \mathbf{a}$, we obtain

$$\begin{cases} \mathbf{a}_{\mathrm{L}} = \mathbf{T}_{\mathrm{MB}}^{\mathrm{T}} \cdot \mathbf{p}_{\mathrm{L}} = \mathbf{T}_{\mathrm{MB}}^{\mathrm{T}} \cdot S_{\mathrm{L}}(\lambda) \cdot \mathbf{p} = \left(\mathbf{T}_{\mathrm{MB}}^{\mathrm{T}} \cdot S_{\mathrm{L}}(\lambda) \cdot \mathbf{T}_{\mathrm{BM}}^{\mathrm{T}}\right) \cdot \mathbf{a} = \mathrm{A}_{\mathrm{L}} \cdot \mathbf{a} \\ \mathbf{a}_{\mathrm{R}} = \mathbf{T}_{\mathrm{MB}}^{\mathrm{T}} \cdot \mathbf{p}_{\mathrm{R}} = \mathbf{T}_{\mathrm{MB}}^{\mathrm{T}} \cdot S_{\mathrm{R}}(\lambda) \cdot \mathbf{p} = \left(\mathbf{T}_{\mathrm{MB}}^{\mathrm{T}} \cdot S_{\mathrm{R}}(\lambda) \cdot \mathbf{T}_{\mathrm{BM}}^{\mathrm{T}}\right) \cdot \mathbf{a} = \mathrm{A}_{\mathrm{R}} \cdot \mathbf{a} \end{cases}$$

Example 2. For $\bar{\mathbf{a}}^{\mathrm{T}} = \begin{bmatrix} -17/90 & 59/36 & 7/9 & -97/36 & -4/45 & 5/9 \end{bmatrix}$ the polynomial $P_n(x)$, given by (3.1) is defined on $[a, b] = [-2,\ 2]$. The coefficients of the polynomial given by (3.2) are

$$\mathbf{a}^{\mathrm{T}} = \begin{bmatrix} 2 & 809/15 & -7112/15 & 58288/45 & -65024/45 & 5120/9 \end{bmatrix}$$

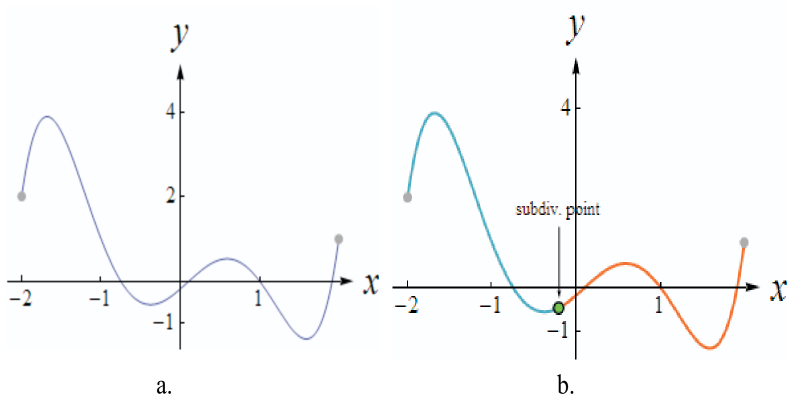

FIGURE 2. Binary subdivision of monomial basis

The subdivision factor is $\lambda = 0.45$ and, according to (3.4), one has

$$\mathbf{a}_{\mathrm{L}}^{\mathrm{T}} = \left[\begin{array}{cccccc} 2 & 24.27 & -96.012 & 118.033 & -59.2531 & 10.4976 \end{array}\right],$$
$$\mathbf{a}_{\mathrm{R}}^{\mathrm{T}} = \left[\begin{array}{cccccc} -0.46432 & 2.22581 & 11.2707 & -25.567 & -15.0965 & 28.6313 \end{array}\right],$$

which implies $P_n^{\mathrm{L}}(x) = \mathbf{a}_{\mathrm{L}}^{\mathrm{T}} \cdot \mathbf{m}_{\mathrm{L}}(x)$, $x \in [a, c]$ and $P_n^{\mathrm{R}}(x) = \mathbf{a}_{\mathrm{R}}^{\mathrm{T}} \cdot \mathbf{m}_{\mathrm{R}}(x)$, $x \in [c, b]$. Of course, as it is expected,

$$P_n^{\mathrm{L}}(x) \equiv P_n^{\mathrm{R}}(x) \equiv -\frac{17}{90} + \frac{59}{36}x + \frac{7}{9}x^2 - \frac{97}{36}x^3 - \frac{4}{45}x^4 + \frac{5}{9}x^5$$

4. The attractor

The subdivision matrices in all above cases define linear mappings in a higher dimensional space $\mathbb{R}^{m+1}$, which means the AIFS. The Hutchinson operator (1.3) can be rewritten as

$$W'(\mathbf{x}) = \bigcup_{i=1}^{n} \mathrm{S}_i(\mathbf{x})$$

Then, the random-type algorithm ([1], [2]) is easy to establish to calculate the orbit of W', i.e., the set $(W')^{\circ m}(\mathbf{x}_0)$, where $\mathbf{x}_0 \in \mathbb{R}^{m+1}$. If we employ the Bernstein polynomial from Example 1, and use the random algorithm with 200 iterations, the result is shown in Fig. 3a Using 1000 iterations gives a more complete picture (Fig. 3b).

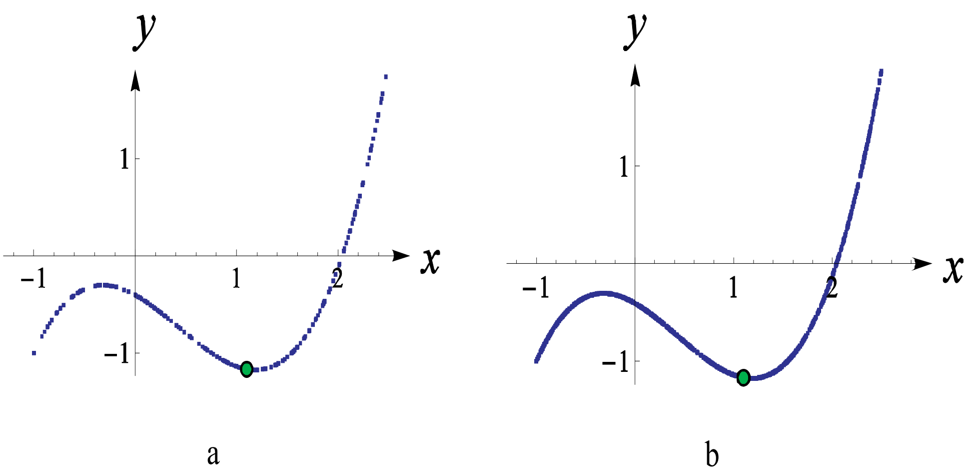

FIGURE 3. Bernstein polynomial as an attractor. a. 200 points; b. 1000 points.

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