THE OPTIMAL PROFIT REPAIRS POLICY OF ONE-COMPONENT MULTI - STATE SYSTEM WITH GRADUATE FAILURE

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Abstract

We regard a multi-state one component system with graduate failure. The one level transitions are homogeneous Markov transitions and the unit time work in the state i makes profit C_i. We suppose that the system runs during n periods with constant length T and after each period we are able to make an inspection of the system. Depending of its current state we can decide to repair it or not. The problem is to find the optimum policy for repairing such that the summary profit be the greatest one.

I. Introduction

A multi-state one component system with graduate failure is a system consist of one component that has $N+1$ states, $0, 1, 2, ..., N$. The prefect state is N and the state 0 is a state of total failure. Greater state means better performance. The graduate failure means that the system can fall down only for one state in a time. We suppose that one level transitions are homogeneous Markov transitions, which imply that one level failure time have an exponential distribution. Let failure intensity from state i to state $i-1$ be λ_i. So, the random variable T_i, the time in which the system constantly works in state i, has an exponential distribution with parameter λ_i. Additionally we suppose that the operations in specifics level i for a unit time result with some constant profit C_i. It is natural to assume that the upper level gives more profit. Such system is given in Figure1.

Such system runs during n periods with constant length T. During each constant period the system can or cannot be repaired. After each period we are able to make an inspection and we can make one of two decisions: to repair the system to its perfect state, which costs some amount R, or to leave it to run from its current state for the next constant period T. Our goal is to find the best practice for recovering this system in a state which leads to maximum profit. In the second section we will give a dynamic programming algorithm that gives the optimal policy that maximizes the total profit. This algorithm is based on the assumption that the profit $\hat{C}_i$ that the system makes during the time T, under assumption that in the beginning of this time period the system is in state i, and the probability $P_{i,j}(T)$, that the system after a time T will be found in state j starting from state i, are known. But in our problem we assume that we known only the one level intensities and the profit that system makes working in some specific level for a unit time. So, in the next section we will give a way for computing $\hat{C}_i$ from C_i and λ_i.

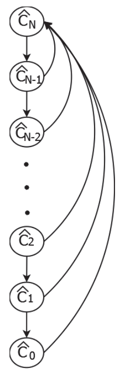

Fig. 1. Visual representation of a system

II. Algorithm for Optimal Repairing Policy

In this section we assume that for each $i = \overline{1,N}$, $\hat{C}_i$, the expected profit under assumption that the system works time T starting in state i, $P_{i,j}(T)$ and repair costs R are known. Let $B[i,k]$ represent profit which we can gain in k-th period if the machine is in i-th state. After each period we calculate whether it is better to repair the machine or not. This decision depends on the previous state and probability of falling. The problem can be considered in aspects of dynamic programming R repair cost and N number of states. If the system before n-th period is in state i, the expected profit if it is repeated is equal to $-R + C_N$, and the expected profit if it is not repaired is C_i. Similarly, if just before the k-th period, the system is in state i, the expected profit if it is repaired is:

$$B_{i,k} = -R + \hat{C}_N + \sum_{r=1}^{N} P_{N,r} * B_{r,k-1}, \qquad (1)$$

and if it is not repaired is

$$B_{i,k} = \hat{C}_i + \sum_{r=1}^{N} P_{i,r} * B_{r,k-1}) \qquad (2)$$

We will do the action that gives grater profit, so the recursive formula is:

$$B_{i,j} = \begin{cases} B_{i,j} = \hat{C}[j], i = 1 \\ B_{i,j} = Max(-R + \hat{C}[N] + \\ \sum_{k=1}^{N} P_{N,k} * B_{i-1,k}, \\ \sum_{k=1}^{N} P_{j,k} * B_{i-1,k}), i > 0, N > j > 0 \end{cases} \quad (3)$$

We will explain the algorithm trough following example.

Example 1. Let us have a system with 5 states. $\hat{C}_i = i$ and the matrix $P(T)$ is given in Figure II. We suppose that the machine starts with its work at its best state, 4.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0.25 & 0.25 & 0.5 & 0 & 0 \\ 0.125 & 0.125 & 0.25 & 0.5 & 0 \\ 0.0625 & 0.0625 & 0.125 & 0.25 & 0.5 \end{bmatrix}$$

So before n-th period we have those situations

- In state 4 the machine should not be repaired. We have a profit of 4.
- In state 3 the machine should not be repaired. We have a profit of 3.
- In state 2 the machine should not be repaired. We have a profit of 2.
- In state 1 the machine should not be repaired. We have a profit of 1.
- In state 0 the machine doesn't work, which means no profit.

Before N-1-th period

- In state 4:
  - We have profit of 7.0625
- In state 3:
  - If we choose to repeat the machine we have profit of 3.0625
  - If we choose to not repeat the machine we have profit of 4.125
- In state 2:
  - If we choose to repeat the machine we have profit of 3.0625
  - If we choose to not repeat the machine we have profit of 3.25
- In state 1:
  - If we choose to repeat the machine we have profit of 3.0625
  - If we choose to not repeat the machine we have profit of 1.5
- In state 0 the machine doesn't work, which means no profit.

and so on.

After five periods the profit matrix and the repair matrix will be

According to repair matrix in the first period we don't have to repair the machine (no matter in what condition it is). In the second period if the machine is in state 2, 3 or 4 we don't have to repair, but if the machine is in the state 1 it is more affectively to repair the machine. At the forth period (and all upper periods) it isn't necessary to repair the machine in state 4, but in all other states we should repair the machine.

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 3.0625 & 3.0625 & 3.25 & 5.125 & 7.0625 \\ 5.60156 & 5.60156 & 5.60156 & 7.14063 & 9.60156 \\ 7.98633 & 7.98633 & 7.98633 & 9.37109 & 11.9863 \\ 10.3325 & 10.3325 & 10.3325 & 11.6787 & 14.3325 \\ 12.6691 & 12.6691 & 12.6691 & 14.0056 & 16.6691 \end{bmatrix}$$

Fig. 2. Profit matrix

Period \ State	0	1	2	3	4
0	0	0	0	0	0
1	1	1	0	0	0
2	1	1	1	0	0
3	1	1	1	0	0
4	1	1	1	0	0

Fig. 3. Repair table (This table show the optimal repair of the system observed in 4 period. 1 means that if we are in state s in period p we should repair system. If there is 0 we should not repair.)

Example 2. There is a example how the system works with repair cost equal to 6, $\hat{C}_i = \{0, 1, 2, 3, 4\}$ where i is a state and next Probability matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.8 & 0.2 & 0 & 0 & 0 \\ 0.7 & 0.2 & 0.1 & 0 & 0 \\ 0.6 & 0.25 & 0.15 & 0.1 & 0 \\ 0.5 & 0.2 & 0.15 & 0.1 & 0.05 \end{bmatrix}$$

The results are:

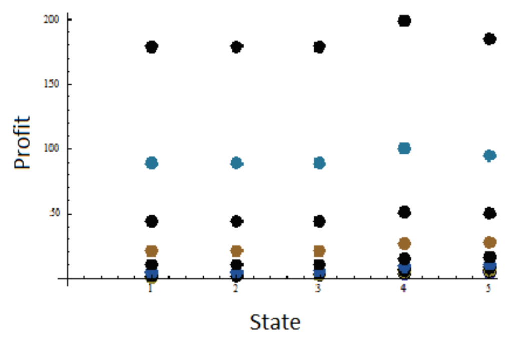

Fig. 4. Profit of a system (For all states we display expected profit after some period of time

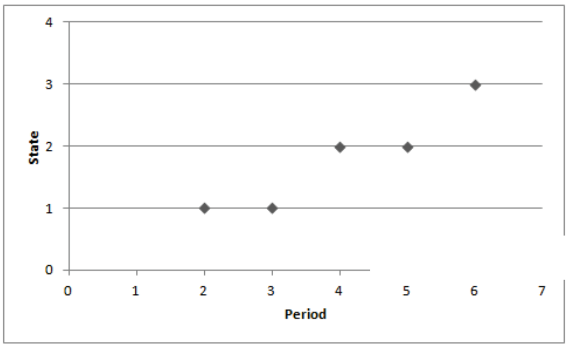

Fig. 5. Repair of a system (The method for repair the system in order to maximise expected profit. Point on state 3, period 5 means that if system is in state 3 or below in period 5 the system should be repaired

Example 3. Another example with repair cost equal to 10, $\hat{C}_i = \{0, 1, 2, 10, 20\}$ where i is a state and same Probability matrix as in Example 2.

The results are:

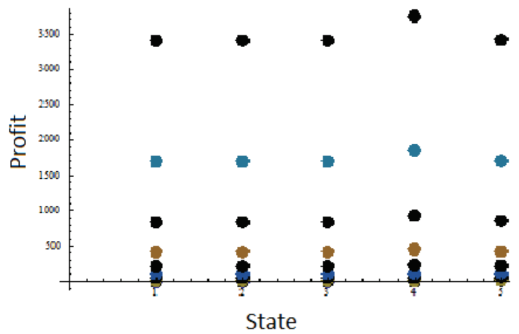

Fig. 6. Profit of a system

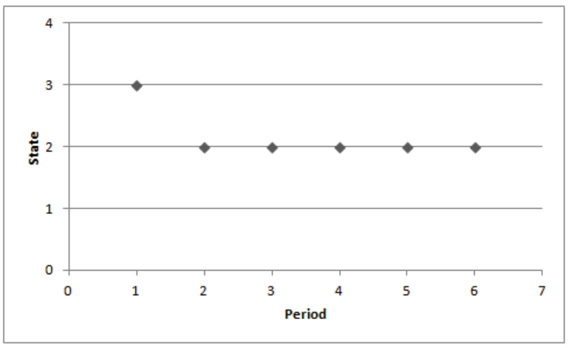

Fig. 7. Rapeir of a system

The pseudo code for algorithm is following:

for $i = 1 \to time$ **do**
  for $j = 1 \to N$ **do**
    $sr \leftarrow 0$
    if $j < N$ **then**
      for $k = 1 \to N$ **do**
        $sr \leftarrow sr + rm[i, k] * P[N, k]$
      end for
    end if
    $ssr \leftarrow sr - R + rm[i, N]$
    $snr \leftarrow 0$
    for $k = 1 \to N$ **do**
      $snr \leftarrow snr + rm[i, k] * P[j, k]$
    end for

    $snr \leftarrow snr + rm[i, j]$
    [illegible]
      [illegible]
      $rem[i + 1, j] \leftarrow 1$
    else
      $rm[i + 1, j] \leftarrow snr$
      $rem[i + 1, j] \leftarrow 0$
    end if
  end for
end for

III. Computation of the Expected Profit for System with Independent One-level Transitions

It is naturally to assume that the expected profit under assumption that the system works time T starting in state i is unknown, but we know the profit system makes when it is working for a unit time in a certain level. Moreover, we will suppose that the one level transitions are monotone Markov transitions with known one level failure intensities. From the other hand, to calculate the optimal repair policy we need tho now the probability that the system that starts its work in level i after time T will found in state j, i.e. the matrix P. Therefore, the problem we analyze here is finding the unknown values for the given ones.

Since one level transitions are monotone Markov transitions, the distribution of time of work in the i-th level has exponential distribution and the probabilities $P_{i,j}$ when all the intensities are equal can be calculate using next Theorem:

Theorem 1: Suppose that a multi-state one component system with graduate failure and equal transition intensities, λ works time T starting in state i. Then the probability that a system will finish its work in state j is:

$$P_{i,j}(t) = \frac{\lambda^{i-j} t^{i-j}}{(i-j)!} e^{-\lambda t}, \qquad (4)$$

The proof of the theorem is given in [2], where it can be found the formula for calculation same probabilities for system with different failure intensities.

In order to find formula for calculation of the expected profit, we use following notations: C_i - the profit system makes when it works with level i a unit period without failure, and $\widehat{C}_i(T)$ - the profit system makes when works a time T without reparation, starting at level i. Next theorem gives a formula for calculation of $\widehat{C}_i(T)$.

Theorem 2: Suppose that a multi-state one component system with graduate failure and one level transition intensities λ_k works time T starting in state i. Than the expect cost $\widehat{C}_i(T)$ can be calculate by:

$$\hat{C}_i(T) = \int_{t_i=T}^{\infty} C_i T e^{-\lambda_i t_i}\,\mathrm{d}t +$$
$$\int_{t_i=0}^{T} \int_{t_{i-1}=T-t_i}^{\infty} (C_i t_i + C_{i-1}(T-t_i)) e^{-\lambda_{i-1} t_{i-1}} e^{-\lambda_i t_i} dt_{i-1} dt_i$$
$$+ \ldots + \int_{t_i=0}^{T} \int_{t_{i-1}=0}^{T-t_i} \ldots \int_{t_1=T-(t_i+\ldots+t_2)}^{\infty} (C_i t_i + C_{i-1} t_{i-1}$$
$$+ \ldots + C_2 t_2 + C_1(T - \sum_{k=2} t_i)) e^{-\lambda_i t_i} \ldots e^{-\lambda_1 t_1} \mathrm{d}t_1 \ldots \mathrm{d}t_i$$
$$+ \int_{t_i=0}^{T} \int_{t_{i-1}=0}^{T-t_i} \ldots \int_{t_1=0}^{T-(t_i+\ldots+t_2)} (C_i t_i + C_{i-1} t_{i-1} + \ldots +$$
$$C_1 t_1 + C_0(T - \sum_{k=1} t_i)) e^{-\lambda_i t_i} \ldots e^{-\lambda_0 t_0} \mathrm{d}t_1 \ldots \mathrm{d}t_i \tag{5}$$

Proof: Let A_i be the event: The system works time T without reparation, starting from the state i; and B_j be the event: At the and of the work, the system is in the state j. Then,

$$P(A_i) = \sum_{j=0}^{i} P(A_i B_j).$$

It is clear that $P(A_i B_j)$ is

$$\int_{t_i=0}^{T} \int_{t_{i-1}=0}^{T-t_i} \ldots \int_{t_j=T-\sum_{k=j+1}^{i} t_k}^{\infty} \left(\left(\sum_{k=j+1}^{i} C_k t_k \right) + \right.$$
$$\left. C_j(T - \sum_{k=j} t_i) \right) e^{-\lambda_i t_i} \ldots e^{-\lambda_j t_j} \mathrm{d}t_j \ldots \mathrm{d}t_i.$$

The Theorem follows directly from (5). ■

The last Theorem gives complicate expression for calculation of $\widehat{C}_i(T)$, so next we will give recursive relationship between the integrals found in that equation.

Let by $D_k(T, C_k, C_{k-1}, ..., C_1)$ be the k-th integral of (5) where the constant a set on C_k, C_{k-1}, ...,C_1 on the same order as that found in the integral. Then

$$D_1(T, C_1) = \int_{t_1=T}^{\infty} C_1 T e^{-\lambda_1 t_1} \mathrm{d}t_i.$$

The integral $D_2(T, C_2, C_1)$ can be expressed using $D_1(T - t_1, C_1)$ as

$$\int_{t_2=0}^{T} \int_{t_1=T-t_2}^{\infty} (C_2 t_2 + C_1(T-t_2)) e^{-\lambda_1 t_1} e^{-\lambda_2 t_2} \mathrm{d}t_1 \mathrm{d}t_2 =$$
$$\int_{t_2=0}^{T} \int_{t_1=T-t_2}^{\infty} C_2 t_2 e^{-\lambda_1 t_1} e^{-\lambda_2 t_2} \mathrm{d}t_1 \mathrm{d}t_2 +$$
$$\int_{t_2=0}^{T} (\underbrace{\int_{t_1=T-t_2}^{\infty} C_1(T-t_2) e^{-\lambda_1 t_1} \mathrm{d}t_1}_{D_1(T-t_2, C_1)}) e^{-\lambda_2 t_2} \mathrm{d}t_2 =$$
$$G_2(C_2, T) + \int_{t_k=0}^{T} D_1(T - t_2, C_1) e^{-\lambda t_2} \mathrm{d}t_2.$$

Similarly,

$$D_k(T, C_k, \ldots, C_1) =$$
$$\int_{t_k=0}^{T} \ldots \int_{t_2=0}^{T-\sum_{r=3}^{k} t_r} \int_{t_1=T-t_2-\ldots t_k}^{\infty} C_k t_k e^{-\sum_{r=3}^{k} t_k \lambda_r t_r} \mathrm{d}t_1 \ldots \mathrm{d}t_k$$
$$+ \int_{t_k=0}^{T} D_{k-1}(T - t_k, C_{k-1}, \ldots, C_1) e^{-\lambda t_k} \mathrm{d}t_k =$$

1 $$G_k(C_k, T) + \int_{t_k=0}^{T} D_{k-1}(T - t_k, C_{k-1}, \ldots, C_1) e^{-\lambda t_k} \mathrm{d}t_k.$$

We will denote the last integrals by $\widehat{D}_i(T, C_1, \ldots, C_n)$, i.e.

$$\hat{D}_k(T, C_1, \ldots, C_k) = \int_{t_i=0}^{T} \int_{t_{i-1}=0}^{T-t_i} \ldots \int_{t_1=0}^{T-(t_i+\ldots+t_2)} (\sum_{r=1}^{i} C_r t_r +$$
$$C_0(T - \sum_{r=1}^{k} t_r)) e^{-\lambda_i t_i} \ldots e^{-\lambda_0 t_0} \mathrm{d}t_1 \ldots \mathrm{d}t_i$$

Now, the expected costs are:

$$\hat{C}_k = \sum_{r=1}^{k} D_r(T, C_r, \ldots, C_{k-r+1}) + \hat{D}(T, C_k).$$

This can be solved in bottom-up principe, again using a dynamic programming algorithm.

IV. Conclusion and Future Work

In this paper we find the optimal policy for repairing (replacing) a one component multi-state system. The management is in respect to profit optimization. The system works for a constant time T and the problem we interested in is, depending to the level in which the system is found after that time, to decide is it better to repair the system or not. First we give a method in which the expected profit, repairing time and transition failure probabilities. Moreover we analyze how to find this unknown values, assuming that the one-level transition intensities, the profit for unit time in certain level and time T are known and we give a recursive formulation of that problem. On the basis of this formula we give a dynamic programming algorithm.

Our next work will be focused on finding the highest level in which system should be repaired in order to get profit. Also we will try to make further simplifications of the given recursive formula.

References

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[2] Marija L. Mihova, *Modelling the reliability of unrecoverable multi-state systems with independent equal components and homogeneous Markov transitions*, Institute of mathematics, Faculty of Natural Science and Mathematics, Skopje, 2008