Int. J. Adv. Appl. Math. and Mech. 6(2) (2018) 14 – 19 (ISSN: 2347-2529)

Journal homepage: www.ijaamm.com

IJAAMM

International Journal of Advances in Applied Mathematics and Mechanics

Mathematical model of relativistic 3-acceleration

Research Article

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Received 12 December 2018; accepted (in revised version) 24 December 2018

Abstract: The standard Newtonian acceleration $\vec{a}$ is one of the rare physical quantities that does not have corresponding relativistic analog. We introduce a relativistic 3-acceleration $\vec{a}_{rel}$ derived directly from relativistic velocity addition law. Actually, $\vec{a}_{rel}$ is shown to be a 3-acceleration in an instantaneously comoving rest frame represented in terms of coordinates of the corresponding 4-acceleration A.

The relativistic 3-acceleration $\vec{a}_{rel}$ possesses some interesting features which enable to express some of the relativistic dynamic quantities in a more convenient way. Particularly, relativistic 3-force takes convenient Newtonian form $\vec{f}_{rel} = m\vec{a}_{rel}$ from which the relativistic formula for energy naturally arises. The general idea behind $\vec{a}_{rel}$ has been implicitly exploited in other papers, commonly through the physical concept of 3-force and the corresponding equation of motions. However, we present a simple mathematical model which gives explicitly the true $\vec{a}_{rel}$ origin and different ways of its derivation in a mathematically more compelling way.

MSC: 83A05 • 70B05

Keywords: Special relativity • 4-velocity • Hyperbolic operations

1. Introduction

The standard Newtonian acceleration is one of the basic quantities in classical mechanics [1]. It appears in the theory of dynamical systems as the second order derivative within equations of motion. For example, in an oscillatory motion, either harmonic [2] or anharmonic [3], this quantity is the main vector variable which relates the forces in the system with the particle position and velocity. Generally, it is also customary to use Newton's laws in cases of gravitational acceleration [4].

The notion of acceleration in the special theory of relativity is usually discussed in terms of 4-acceleration A. However, there is a difficulty to define this notion relativistically in 3D with clear physical meaning and "good behavior" under the Lorentz transformations. Indeed, Lorentz transformations of a 4-acceleration do not reveal any clear kinematic role of the temporal and the spatial part. With exception of the case in the proper frame, the temporal component of a 4-acceleration vector, under Lorentz transformations, enters in the spatial part of the vector, making it complicated to understand.

It is well known that the Einstein's velocity addition law with its hyperbolic nature has a rich structure and hence, it could be placed centrally in the special relativity theory [5]. Neglecting the hyperbolic nature of relativistic velocity operations by accepting Newtonian acceleration $\vec{a}$ as a measure of the change of velocity is inconsistent with the relativity theory, despite the fact that it leaves us with a quite comfortable mathematical apparatus. So, we propose a

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natural way to define a relativistic 3-acceleration $\vec{a}_{rel}$ by considering the rate of change of a relativistic velocity that clearly includes the subtraction $\vec{v}(t+h) \ominus \vec{v}(t)$, or more precisely $\ominus\vec{v}(t) \oplus \vec{v}(t+h)$, where $\oplus$ and $\ominus$ are the relativistic addition and subtraction, respectively. In this way, $\vec{a}_{rel}$ could be interpreted as a new physical quantity with a distinguished physical meaning.

The relativistic acceleration $\vec{a}_{rel}$ provides a tool for derivation of the relativistic dynamic quantities where the relativistic 3-force takes the convenient Newtonian form $\vec{f}_{rel} = m\vec{a}_{rel}$. Indeed, this relativistic force formula naturally leads to the standard expression for total relativistic energy $E = \gamma mc^2$. So, observing from another perspective, $\vec{a}_{rel}$ enables us to avoid somewhat arbitrary introduction of relativistic 3-momentum and to introduce complete relativistic analogs to classical Newtonian laws in 3D notation.

We will also show that a form of $\vec{a}_{rel}$ has been implicitly used by some authors, in their attempts to derive equation of motion based on relativistic analog of the Newtonian second law. Typical examples are [6] and [7] where the expression for relativistic force is derived in a way to mimic the Newtonian second law. However, these approaches do not give a mathematical model to explain the derivation of relativistic 3-acceleration which is only implicitly involved in the presented formulas for relativistic 3-force.

2. Relativistic 3-acceleration

Let us consider 4-velocities in the Minkowski space with the metric with signature $(+,-,-,-)$ and 4-velocities $U = (\gamma_u c, \gamma_u \vec{u})^\top$ and $V = (\gamma_v c, \gamma_v \vec{v})^\top$ where $\vec{u}$, $\vec{v}$ are the corresponding 3-velocities as vector columns and $\gamma_u = \frac{1}{\sqrt{1-\frac{\vec{u}^2}{c^2}}}$, $\gamma_v = \frac{1}{\sqrt{1-\frac{\vec{v}^2}{c^2}}}$ are the corresponding gamma factors. One can easily derive the general formula for relativistic 3-velocity addition $\vec{u} \oplus \vec{v}$ considering two frames moving with relative 4-velocity U. Then, the velocity V in the first frame transforms into velocity W in the second frame by the Lorentz boost $B(\vec{u})$ (see for example [8]) in the following way

$$W = (\gamma_w c, \gamma_w \vec{w})^\top = B(\vec{u})V = \begin{bmatrix} \gamma_u & \dfrac{\gamma_u \vec{u}^\top}{c} \\ \dfrac{\gamma_u \vec{u}}{c} & I_3 + \dfrac{\gamma_u^2}{(1+\gamma_u)c^2}\vec{u}\vec{u}^\top \end{bmatrix} \begin{bmatrix} \gamma_v c \\ \gamma_v \vec{v} \end{bmatrix} = \begin{bmatrix} \gamma_u \gamma_v c(1+\dfrac{\vec{u}\vec{v}}{c^2}) \\ \gamma_u\gamma_v \vec{u} + \gamma_v \vec{v} + \dfrac{\gamma_u^2 \gamma_v (\vec{u}\vec{v})\vec{u}}{(1+\gamma_u)c^2} \end{bmatrix}. \tag{1}$$

It follows $\gamma_w = \gamma_{\vec{u}\oplus\vec{v}} = \gamma_u \gamma_v (1 + \frac{\vec{u}\vec{v}}{c^2})$ and

$$\vec{w} = \vec{u} \oplus \vec{v} = \frac{1}{\gamma_w}\Big(\gamma_u\gamma_v\vec{u} + \gamma_v\vec{v} + \frac{\gamma_u^2\gamma_v}{(1+\gamma_u)c^2}(\vec{u}\vec{v})\vec{u}\Big) = \frac{1}{1+\dfrac{\vec{u}\vec{v}}{c^2}}\Big(\vec{u} + \frac{1}{\gamma_v}\vec{v} + \frac{1}{c^2}\frac{\gamma_u}{1+\gamma_u}(\vec{u}\vec{v})\vec{u}\Big),$$

or using cross product,

$$\vec{w} = \frac{1}{1+\dfrac{\vec{u}\vec{v}}{c^2}}\Big(\vec{u} + \vec{v} + \frac{1}{c^2}\frac{\gamma_u}{1+\gamma_u}\vec{u}\times(\vec{u}\times\vec{v})\Big).$$

The matrix I_3 is the unit 3×3 matrix and $uu^\top$ is the vector outer product.

This velocity addition law contains an indetermination concerning the order of summation which is, generally, non-commutative and non-associative. Namely, it has been shown (see for example [9]) that $\vec{u} \oplus \vec{v} = T(\vec{u},\vec{v})(\vec{v} \oplus \vec{u})$ and $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus T(\vec{u},\vec{v})\vec{w}$ where $T(\cdot,\cdot)$ is a rotation known as Thomas precession. The Thomas precession is a self-map of the ball of all relativistically admissible velocities and it obstructs commutativity and associativity in the relativistic velocity addition.

The acceleration in classical physics is defined as the rate of change of the vector of 3-velocity $\vec{v}$ given by $\vec{a} = \frac{d\vec{v}}{dt}$ with magnitude a. Analogously, in relativistic physics, we can derive acceleration as a rate of change of relativistic 3-velocity with respect to proper time. The change of relativistic 3-velocity is simply calculated using relativistic velocity subtraction

$$\begin{aligned}
\frac{(d\vec{v})_{rel}}{dt} &= \lim_{h\to 0}\frac{\ominus\vec{v}\oplus\vec{v}(t+h)}{h}\\
&= \lim_{h\to 0}\frac{1}{1-\vec{v}(t)\vec{v}(t+h)c^{-2}}\Big(\frac{-\vec{v}(t)+\vec{v}(t+h)}{h}-\frac{1}{h}\frac{\gamma_v}{c^2(1+\gamma_v)}\vec{v}(t)\times(-\vec{v}(t)\times\vec{v}(t+h))\Big)\\
&= \gamma_v^2\frac{d\vec{v}}{dt}+\frac{\gamma_v^3}{c^2(1+\gamma_v)}\vec{v}\times\lim_{h\to 0}\frac{1}{h}\Big(\vec{v}(t)\times(\vec{v}(t)+\frac{d\vec{v}(t)}{dt}h)\Big)\\
&= \gamma_v^2\frac{d\vec{v}}{dt}+\frac{\gamma_v^3}{c^2(1+\gamma_v)}\vec{v}\times(\vec{v}\times\frac{d\vec{v}}{dt})=\gamma_v^2\frac{d\vec{v}}{dt}+\frac{\gamma_v^3}{c^2(1+\gamma_v)}\Big((\vec{v}\frac{d\vec{v}}{dt})\vec{v}-\vec{v}^2\frac{d\vec{v}}{dt}\Big)\\
&= \gamma_v\frac{d\vec{v}}{dt}+\frac{1}{1+\gamma_v}\frac{d\gamma_v}{dt}\vec{v}=\gamma_v\vec{a}+\frac{1}{1+\gamma_v}\dot{\gamma}_v\vec{v},
\end{aligned}\tag{2}$$

where $(d\vec{v})_{rel}$ denotes the limit of relativistic difference.

From

$$\lim_{h\to 0}\frac{\vec{v}(t+h)\oplus(\ominus\vec{v}(t))}{h}=\frac{(d\vec{v})_{rel}}{dt}$$

it follows that the acceleration vector $\frac{(d\vec{v})_{rel}}{dt}$ is independent of the non-commutativity of the hyperbolic operation $\oplus$, and so it is unambiguously defined. Additionally, one can straightforwardly check that

$$\lim_{h\to 0}\frac{\ominus\vec{v}(t+h)\oplus\vec{v(t)}}{h}=\lim_{h\to 0}\frac{\vec{v}(t)\oplus(\ominus\vec{v}(t+h))}{h}=-\frac{(d\vec{v})_{rel}}{dt}.$$

So, we can use ordinary $-$ for a negative acceleration. Now, to obtain relativistic 3-acceleration $\vec{a}_{rel}$ we simply set

$$\vec{a}_{rel}=\frac{(d\vec{v})_{rel}}{d\tau}=\gamma\frac{(d\vec{v})_{rel}}{dt}=\gamma^2\vec{a}+\frac{\gamma}{1+\gamma}\dot{\gamma}\vec{v},\tag{3}$$

where $\gamma=\gamma_v$.

Indeed, one can consider the space part

$$\vec{a}_f=\gamma\vec{a}+\dot{\gamma}\vec{v}\tag{4}$$

of the relativistic 4-acceleration $A=(\gamma\dot{\gamma}c,\gamma^2\vec{a}+\gamma\dot{\gamma}\vec{v})^\top$ (divided by γ) as a candidate to be called relativistic 3-acceleration. So, $\vec{a}_f$ is simply the part of the acceleration vector that corresponds to relativistic 3-force $\vec{f}_{rel}$. It is obviously different from $\vec{a}_{rel}$ we introduced here. However, as we will show later, there is a Lorentz link between the 3-vectors $\vec{a}_{rel}$ and $\gamma\vec{a}_f$.

The relativistic acceleration $\vec{a}_{rel}$ has the following properties:

1. In the limit case, when $\gamma=1$, then $\vec{a}_{rel}=\vec{a}$ i.e. relativistic acceleration is deduced to standard 3-acceleration.
2. In the special case of rectilinear motion, the acceleration $\vec{a}$ is parallel to the velocity $\vec{v}$ and since

   $$\vec{v}(t)\times\vec{v}(t+h)=0,$$

   we simply have $\vec{a}_{rel}=\gamma^3\vec{a}$. On the other hand, when $\vec{a}$ is perpendicular to $\vec{v}$ as in a circular motion, we get $\vec{a}_{rel}=\gamma^2\vec{a}$.
3. There is a simple relation between $\vec{a}_{rel}$ and $\vec{a}$ when a velocity $\vec{v}$ is involved. This extremely useful equation in calculations takes the form $\vec{v}\vec{a}_{rel}=\gamma^3\vec{v}\vec{a}$. The proof is straightforward.
4. From $\gamma>1$, it immediately follows that $\|\vec{a}_{rel}\|>\|\vec{a}\|$.

Observe that the property 2 is an accepted result in relativistic kinematics.

3. Using $\vec{a}_{rel}$ in relativistic dynamics

It seems obvious that the derivation of relativistic dynamics equations in 3D notation is rather peculiar and mathematically not completely consequent. In absence of the concept of relativistic 3-acceleration, there is no ordinary way to make a bridge between the relativistic momentum and force. That is the reason of so many different explanations and approaches in derivation of relativistic dynamics equations.

Introduction of relativistic 3-acceleration gives a clear opportunity to derive relativistic dynamics equations in an analogous Newtonian form, avoiding the concept of relativistic 3-momentum. Namely, one can argue that relativistic

3-momentum is introduced somewhat artificially, without clear theoretical and logical background. Some authors try to avoid this inconsistency by introducing the rest mass or inertial mass in attempt to justify γ in the 3-momentum definition. So, for example, [10] and [11] introduce the relativistic mass, while [12] and [13] do not. The physical justification of γ is not in question, but the mathematical inconsistency in employing γ in relativistic 3-momentum ($m\frac{d\vec{x}}{d\tau} = \gamma m\vec{v}$) and not employing γ in relativistic 3-force ($f = \frac{d\vec{p}}{dt} = m(\dot{\gamma}\vec{v} + \gamma\vec{a})$, seems obvious.

It is common in relativistic dynamics to emphasize ideas of momentum and energy conservation rather than any adaptation of the Newton second law. Of course, we are aware of its important properties like conservation in non-elastic collisions and some advantages of the straightforward jump from 4-momentum to 4-force avoiding 4-acceleration as a concept that is considered as less useful. However, achieving completely Newtonian analog of relativistic equations is also important goal that obviously needs the concept of relativistic 3-acceleration.

It is interesting to note that in contrast to space-time events, the 4-velocity, the 4-momentum and the subsequent derived 4-vectors, the 4-acceleration and the 4-force have only 3 independent components. Actually, in the 4-acceleration $A = (a_0, \gamma\vec{a}_f)^\top$, where $a_0 = \gamma\dot{\gamma}c$ and $\gamma\vec{a}_f = \gamma^2\vec{a} + \gamma\dot{\gamma}\vec{v}$, we have $a_0 = \gamma\vec{a}_f(\frac{\vec{v}}{c})$ and the same is valid for 4-force with a rest mass. So, the complete physical information about the behavior of an accelerating object is included in the spatial part of the corresponding 4-vector. In such circumstances it seems a bit strange to give advantage to the 4-vector representation, although the latter enables to express physical equations in an elegant-covariant way. Thus, the derivation of relativistic dynamics equations using 3-vectors is a competitive way to express the corresponding physical laws. The following derivation is yet another way to write these equations in a consistent Newtonian-like manner:

- Relativistic velocity: $\vec{v}_{rel} = \frac{d\vec{x}}{d\tau} = \gamma\vec{v}$ (Newtonian: $\vec{v} = \frac{d\vec{x}}{dt}$)
- Relativistic acceleration: $\vec{a}_{rel} = \frac{(d\vec{v})_{rel}}{d\tau} = \gamma\frac{(d\vec{v})_{rel}}{dt}$ (Newtonian: $\vec{a} = \frac{d\vec{v}}{dt}$)
- Relativistic force: $\vec{f}_{rel} = m\vec{a}_{rel}$ (Newtonian: $\vec{f} = m\vec{a}$)
- Relativistic energy: $\vec{f}_{rel}\vec{v}_{rel} = \frac{dE_{rel}}{d\tau}$ (Newtonian: $\vec{f}\vec{v} = \frac{dE}{dt}$)
- Total relativistic energy is $E = \int \vec{f}_{rel}\vec{v}_{rel}d\tau = \int m\vec{a}_{rel}\vec{v}dt = m\int\gamma^3\vec{a}\vec{v}dt = m\int c^2\dot{\gamma}dt = \gamma mc^2$

Thus, in addition to the derivation (2), the equation for relativistic energy is an additional justification $\vec{a}_{rel}$ to be referenced as relativistic 3-acceleration. Notice that everywhere m stands for rest mass. Here, we are not interested in (and do not need) the concept of relativistic or changeable mass.

4. Relation between relativistic 3-acceleration $\vec{a}_{rel}$ and 4-acceleration A

What lies behind the acceleration $\vec{a}_{rel}$ and how to find its physical meaning? Let us consider a velocity $\vec{v}$ in an inertial frame. Informally speaking, the nature of relativistic velocity addition $\vec{u} \oplus \vec{v}$ is to transform the velocity $\vec{v}$ in the system moving with relative velocity $\vec{u}$ with respect to the first one. In the case of derivation of $\vec{a}_{rel}$, we calculate the change of velocity $\vec{v}(t)$ using the expression $\ominus\vec{v}(t) \oplus \vec{v}(t+h)$. Interpreting this expression as previously, we can state that it makes transformation of velocity $\vec{v}(t+h)$ in the system moving with relative velocity $\ominus\vec{v}(t) = -\vec{v}(t)$ with respect to the system moving with velocity $\vec{v}(t+h)$. But the frame moving with relative velocity $-\vec{v}(t)$ with respect to the frame where we consider velocity $\vec{v}(t+h)$, when $h \to 0$, actually approaches an instantaneously comoving rest frame. So, it makes sense to examine the quadruple $(0, \frac{(d\vec{v})_{rel}}{d\tau})^\top = (0, \vec{a}_{rel})^\top$ as a possible 4-vector of acceleration in an instantaneously comoving rest frame.

Indeed, transforming the 4-components $(0, \vec{a}_{rel})^\top$ in the system with velocity $\vec{v}$ we obtain

$$B(\vec{v})(0, \vec{a}_{rel})^\top = (\gamma\dot{\gamma}c, \vec{a}_{rel} + \gamma\frac{\dot{\gamma}\vec{v}}{1+\gamma})^\top,$$

where $B(\vec{v})$ is a Lorentz boost. Now, one can straightforwardly show that the above 4-components give exactly the 4-acceleration

$$(\gamma\dot{\gamma}c, \vec{a}_{rel} + \gamma\frac{\dot{\gamma}\vec{v}}{1+\gamma})^\top = (\gamma\dot{\gamma}c, \gamma^2\vec{a} + \gamma^2\frac{\dot{\gamma}\vec{v}}{1+\gamma} + \gamma\frac{\dot{\gamma}\vec{v}}{1+\gamma})^\top = (\gamma\dot{\gamma}c, \gamma^2\vec{a} + \gamma\dot{\gamma}\vec{v})^\top = A$$

Thus, the 3-vector $\vec{a}_{rel}$ is just a 3-acceleration in an instantaneously comoving rest frame represented in terms of coordinates of the corresponding 4-acceleration A. The easiest way to obtain $\vec{a}_{rel}$ is simply to calculate the space part of the 4-vector $B(-\vec{v})A = B(-\vec{v})(\gamma\dot{\gamma}c, \gamma^2\vec{a} + \gamma\dot{\gamma}\vec{v})^\top = (0, \vec{a}_{rel})^\top$.

5. Discussion

Let us consider the space part $\vec{a}_f = \gamma\vec{a} + \dot{\gamma}\vec{v}$ of the 4-acceleration vector A (without γ). The acceleration $\vec{a}_f$ is the standard route to the relativistic energy expression since it arises in derivation of the momentum $\vec{p} = m\gamma\vec{v}$ (derivation of $\gamma\vec{v}$). In this way, one actually uses $\vec{a}_f$ i.e. $\vec{v}\vec{a}_f$ to derive $E = \gamma mc^2$. On the other hand, it is easy to check that $\vec{v}\vec{a}_f = \vec{v}\vec{a}_{rel}$, and so, $\vec{a}_{rel}$ i.e. $\vec{v}\vec{a}_{rel}$ is another way to derive $E = \gamma mc^2$. The route toward the relativistic energy via $\vec{a}_{rel}$ is mathematically more competitive compared to the standard route, since $\vec{a}_{rel}$ is obtained as a natural relativistic change of velocity given by $(d\vec{v})_{rel}$ (see (2)).

A similar attempt to rediscover a Newtonian second law in special relativity is given in [6]. For that purpose, the authors have introduced 3-force vectors $\vec{f}_{rest}$ (the author used $\vec{F}_r$) in an instantaneously rest frame with the rest mass of the form

$$\vec{f}_{rest} = (\frac{d\vec{p}}{dt})_{\parallel\vec{v}} + \gamma(\frac{d\vec{p}}{dt})_{\perp\vec{v}} \tag{5}$$

where indices denote the components of $\frac{d\vec{p}}{dt}$ parallel and perpendicular to the velocity $\vec{v}$ respectively. However, he lacks to give any explicit expression for $\vec{f}_{rest}$ or for the involved acceleration in the expression (5). To avoid unnecessary carrying of rest mass through equations, we will use $\vec{a}_f = \gamma\vec{a} + \dot{\gamma}\vec{v}$ instead of $\frac{d\vec{p}}{dt}$. So, in our notation, (5) is the 3-acceleration in an instantaneously rest frame of the form $\vec{a}_{rest} = (\vec{a}_f)_{\parallel\vec{v}} + \gamma(\vec{a}_f)_{\perp\vec{v}}$. To develop further the results in [6], we will give the vector $\vec{a}_{rest}$ explicitly and show that it is exactly $\vec{a}_{rel}$ when expressed in terms of the components of 4-acceleration A. The transformation of $\vec{a}_f$ from the system in rest to the system with velocity $\vec{v}$ (for example see [14]) is given by

$$\vec{a}_f = \frac{\vec{a}_{rest}}{\gamma} + \frac{\vec{v}\vec{a}_{rest}}{\vec{v}^2}\vec{v}(1 - \frac{1}{\gamma}) \tag{6}$$

and now $(\vec{a}_f)_{\parallel\vec{v}} = (\vec{a}_{rest})_{\parallel\vec{v}}$ and $(\vec{a}_f)_{\perp\vec{v}} = \frac{1}{\gamma}(\vec{a}_{rest})_{\perp\vec{v}}$ that straightforwardly implies $\vec{a}_{rest} = (\vec{a}_f)_{\parallel\vec{v}} + \gamma(\vec{a}_f)_{\perp\vec{v}}$. Now we are able to express $\vec{a}_{rest}$ through $\vec{a}_f$ in the following way

$$\vec{a}_{rest} = \gamma\vec{a}_f + \frac{\vec{v}\vec{a}_f}{\vec{v}^2}\vec{v}(1-\gamma) \tag{7}$$

which can be verified by inserting (7) in the right-hand side of (6). To express $\vec{a}_{rest}$ in terms of the coordinates of the corresponding 4-acceleration A, we simple replace $\vec{a}_f = \gamma\vec{a} + \dot{\gamma}\vec{v}$ and obtain

$$\vec{a}_{rest} = \gamma^2\vec{a} + \gamma\dot{\gamma}\vec{v} + (1-\gamma)\dot{\gamma}\vec{v} + (1-\gamma)\frac{\gamma}{\vec{v}^2}(\vec{v}\vec{a})\vec{v} = \gamma^2\vec{a} + \frac{\gamma}{1+\gamma}\dot{\gamma}\vec{v} = \vec{a}_{rel} \tag{8}$$

i.e. the relativistic acceleration $\vec{a}_{rest}$ is exactly $\vec{a}_{rel}$ when it is expressed in terms of $\vec{a}_f$. Thus, $\vec{a}_{rel} = (\vec{a}_f)_{\parallel\vec{v}} + \gamma(\vec{a}_f)_{\perp\vec{v}}$ is yet another, more implicit way, to arrive to relativistic 3-acceleration $\vec{a}_{rel}$.

Some aspects of deriving the equation of motion using Newtonian second law has been discussed in [7]. He uses different expressions for different type of forces: gravitational, mechanical and electromagnetic. His equation of motion for gravitational force, again represented through acceleration by excluding rest mass, is of the form

$$\frac{d}{dt}(\gamma\vec{v}) = \vec{a}_f = \frac{1}{\gamma}\Big(\vec{a}_{rest1} + (\gamma-1)\frac{\vec{v}\vec{a}_{rest1}}{\vec{v}^2}\vec{v}\Big)$$

i.e. working similarly as we arrived to (6), we obtain

$$\vec{a}_{rest1} = \gamma(\gamma\vec{a}_f + \frac{\vec{v}\vec{a}_f}{\vec{v}^2}\vec{v}(1-\gamma))$$

and replacing $\vec{a}_f = \gamma\vec{a} + \dot{\gamma}\vec{v}$ gives

$$\vec{a}_{rest1} = \gamma(\gamma^2\vec{a} + \frac{\gamma}{1+\gamma}\dot{\gamma}\vec{v}) = \gamma\vec{a}_{rel}.$$

Thus, in this formulation, relativistic acceleration $\vec{a}_{rel}$ appears once more implicitly as a cornerstone of the equations of motion.

6. Conclusions

We introduced a concept of relativistic 3-acceleration $\vec{a}_{rel}$ and gave it a reasonable mathematical and physical meaning. Indeed, $\vec{a}_{rel}$ is not a new relativistic quantity in mathematical sense, but rather a new interpretation of a quantity with recognizable "good" mathematical behavior. The used term "relativistic 3-acceleration" for the acceleration $\vec{a}_{rel}$ is justified by the following arguments:

- $\vec{a}_{rel}$ can be obtained as the rate of change of relativistic 3-velocity $(d\vec{v})_{rel} = \ominus\vec{v} \oplus \vec{v}(t+h)$
- The relativistic 3-force takes the standard Newtonian form $\vec{f}_{rel} = m\vec{a}_{rel}$, from which relativistic energy immediately arises $E = \int \vec{f}_{rel}\vec{v}dt = \gamma mc^2$.
- Expressed in terms of components of 4-acceleration, $\vec{a}_{rel}$ is just a 3-acceleration in an instantaneously comoving rest frame.

This mathematical model of relativistic 3-acceleration can be used to represent various physical quantities related to the relativistic dynamics and equations of motion in mathematically more consistent way .

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